



**SELECTION AND APPLICATION  
OF DISTORTED RISK MEASURES**

THESIS

Edwin J. Offutt, Major, USAF

AFIT/GOR/ENS/05-14

**DEPARTMENT OF THE AIR FORCE  
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**Wright-Patterson Air Force Base, Ohio**

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THESIS

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## **Abstract**

This thesis develops and illustrates a methodology for the selection of probability distributions and distortion functions associated with risk scenarios resulting from military capability shortfalls. Distorted (or transformed) risk measures are analyzed and applied to account for loss scenarios that may occur with low frequency but result in catastrophic outcomes. After reviewing the rudimentary concepts of distortion, four well-known continuous distributions, suitable for modeling risk scenarios, are chosen using defined criteria. Based on subject matter expert inputs, a simple method for assigning exactly one of the four distributions to any risk scenario is proposed. Four parametric distortion functions from the finance and insurance literature are then selected and applied to each of the chosen distributions. The distortion effects are examined analytically, graphically, and empirically, and broad-based recommendations are recorded as to the instances when one distortion function might be preferred over others. An end-to-end notional problem – in which a subset of available mitigation measures are selected to counteract a multi-faceted risk environment – illustrates the means by which the proposed methodology may be used to affect future systems acquisition through the Capabilities Review and Risk Assessment (CRRA) process of the United States Air Force.

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Edwin J. Offutt

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# SELECTION AND APPLICATION OF DISTORTED RISK MEASURES

## 1. Introduction

International politics and conflict have seen a period of tremendous upheaval since the breakup of the Soviet Union. In that time, the U.S. Air Force (USAF), a fulcrum of the U.S.'s position as the world's preeminent military power, has necessarily shifted its focus to include global operations of virtually every conceivable type, from combat to humanitarian efforts to satellite communications. Recognizing this wide range of requirements, the USAF has released a planning document entitled *The U.S. Air Force Transformation Flight Plan* [8]. From the Foreword, the document states that

... new national security realities have forced us to redefine our enemies as well as our concepts of defense. ... America's armed forces must be re-balanced for future operations. What we require is a capability mix consistent with pre-defined operational concepts and effects-driven methodology. Future programs must be conceived with this mix in mind. Systems or capabilities based on arguments that do not consider the emerging joint character or the asymmetric nature of warfare will find themselves obsolete, irrelevant, and candidates for elimination.

The desirable systems referred to in this quote take advantage of the rapid advances in materials and information technologies that greatly increase the systems' capabilities but also their expense, complexity, and production times. With new global operations strapping the Department of Defense (DoD) budget, there is little room for error when making decisions regarding weapons systems acquisitions; the USAF needs to derive substantial benefit from every weapons system it acquires and operates.



## **1.1 Background**

To better manage escalating weapons systems' cost, complexity, and production time, the acquisitions process has also changed. Rather than focusing on all the things a new weapons system can do, decisions are to be based on filling the gaps where USAF capability is lacking. To aid in locating the gaps (or shortfalls), the Air Force has developed six Concepts of Operations (CONOPS). These are simply the general areas in which the service desires proficiency: Global Mobility, Global Response, Global Strike, Homeland Security, Nuclear Response, and Space and Command, Control, Communications, Computers, Intelligence, Surveillance, and Reconnaissance (C4ISR) [8:61].

In order to precisely assess each CONOPS, the Capabilities Review and Risk Assessment (CRRA) identifies and analyzes current and future capabilities, capabilities' shortfalls, health, risks, and opportunities. The CRRA is a two-fold process: each CONOPS executes a CRRA within its effects and capability purview. Then, an integrated CRRA assesses capabilities and capability shortfalls across all CONOPS. The CONOPS first identify desired warfighting effects and then develop top-level capabilities required to generate those effects. The CRRAs then identify capability gaps, overlaps, and robustness within each top-level capability. Finally, the Integrated CRRA identifies an acceptable level of risk and risk mitigation measures within each capability. This assessment helps the CONOPS Champions articulate any disconnects between required capabilities and programs [8:47].

The USAF's list of desired warfighting effects and top-level capabilities is called the Master Capabilities Library (MCL). The following is a list of the nine top-level capabilities which the USAF examines via the CRRA process.

1. *Surveillance and Reconnaissance.* The capability to successfully conduct surveillance and reconnaissance missions to satisfy Commanders' Priority Intelligence Requirements (PIRs).

2. *Intelligence.* An integrated capability to provide accurate, timely information and thereby achieve the Predictive Battlespace Awareness (PBA) required to plan and conduct operations.
3. *Command and Control.* The exercise of authority and direction by a properly designated commander over assigned and attached forces in the accomplishment of the mission.
4. *Communications.* The ability to represent transfer, compute, and assure data among persons and machines.
5. *Force Application.* Capability to survive and engage a variety of targets throughout the battlespace by kinetic (nuclear and non-nuclear) and non-kinetic means.
6. *Force Projection.* The ability to project and extend national power (military and non-military) around the globe in a timely manner.
7. *Protect.* The integrated application of offensive and defensive actions that detect, assess, predict, warn, deny, respond, and recover, preempt, mitigate, or negate from threats against or hazards to air and space operations, critical infrastructure, and assets, and personnel based on an acceptable level of risk.
8. *Prepare and Sustain.* Activities required to establish operating locations, generate the mission, support and sustain the mission, and posture responsive forces.
9. *Create the Force.* Organize, train, and equip the combat and support capabilities of the Total Force to meet global combatant commander requirements. Maintain sufficient capacities of created forces. [1]

Each capability is further divided into sub-capabilities to aid in identification of “gaps, overlaps, and robustness.” The opinions of subject matter experts (SMEs) are required in assessing where the Air Force currently stands in relation to the desired capabilities of the MCL (i.e., in assessing the current levels of risk). Evaluating the

risk reduction alternatives, however, can be done objectively once the SMEs have contemplated the scenarios and recorded their opinions.

To standardize the method SMEs use within the CRRA to express their opinions, the USAF defined eight *severity factors* and crossed them in a table with six *consequence categories*. The severity factors are “macro-vulnerabilities” which must be considered during hostilities with another nation or entity:

1. Achievement of (military) objectives
2. Friendly casualties
3. Friendly (military) capabilities
4. Friendly (homeland) infrastructure
5. Collateral damage (over all geographic areas)
6. Enemy escalation, to include weapons of mass destruction (WMD)
7. U.S. national integrity
8. U.S. government function.

The ordered consequence categories include *minor*, *modest*, *substantial*, *major*, *extensive*, and *catastrophic*; forty-eight separate, verbal definitions are provided in the table, one for each distinct cross-reference (e.g., a *major* compromise in *U.S. national integrity*) [27].

As described above, a CRRA team is assembled for each of the nine top-level capabilities. For shortfalls within that specific capability, the team assigns a weight to all eight of the severity factors, indicating the likelihood of the worst consequence coming from that particular severity factor. All of the assigned weights are like probabilities in that they must sum to unity. Assuming, one by one, that each of the severity factors is realized, a consequence score of 1 (minor) to 6 (catastrophic) is assigned to each particular severity factor. The *severity score* for the entire CRRA

top-level capability is then the sum of the individual severity factor probabilities times their associated consequence scores.

This description of the CRRA risk analysis process highlights an important aspect of any mathematical description of risk: it is typically defined as the probability of a negative consequence in conjunction with the “harshness” (severity) of the consequence. (This thesis will not consider situations in which there are “positive” or “desirable” consequences.) This definition is directly in line with the concept of expected value from probability and statistics; as a matter of fact, mathematical expected value is a popular risk measure.

Of course, in any subjective assessment of risk there is almost always some uncertainty. If twenty SMEs were asked to assess the risk involved in the service’s current position relative to the highest standard defined by any one of the MCL’s top-level capabilities, there might exist twenty distinct opinions, depending on the degree of precision in the response scale. The range of opinions of the SMEs could, therefore, be said to be probabilistically distributed over some (hopefully narrow) interval, so that the service could have some quantifiable degree of confidence in the risk mitigation measures it enacts. Reasonably, then, a case can be made for considering a *continuous* aspect to risk, as opposed to a *discrete* one.

## **1.2 Problem Definition**

As previously established, the mathematical formulation for the CRRA involves assigning a single, discrete point value to the risk assessments. Conversely, Woodward [32], in his examination of the CRRA process, proposed a method for assigning continuous *probability distributions* to the opinions of the SMEs, and then described four different risk measures that can be used to summarize the assigned distribution. Making some key simplifying assumptions, the risk measure he recommends for the CRRA process is the expected value of a “distorted” risk distribution.

The selected risk measure is subsequently used to establish the value of potential risk mitigators (e.g., new weapons systems) in terms of the ranked capability gap risks, and then to solve a mathematical program which selects a subset of the potential acquisitions. Given a few simplifying assumptions, Woodward [32] presents a complete, top-to-bottom methodology for the CRRA process.

However, many questions remain unanswered in the risk analysis literature. First, for the CRRA process and many other similar applications, what comprises a reasonable set of continuous distributions from which to choose? While the exponential and Weibull distributions are popular and practical choices for many applications in the fields of risk analysis and reliability engineering (see, for instance, [9] and [11]), other distributions should also be considered if one desires a more versatile set of choices for describing a specific risk scenario. For instance, the exponential and Weibull distributions can only model a nonnegative random variable; other probability distributions may offer capabilities which complement those two distributions.

Second, if a distortion function is used to transform the risk distribution and therefore change the risk measure(s), which distortions should be considered? The concept of “distorting” (or transforming) a risk distribution is a relatively recent development originating in the field of insurance and financial portfolio risk analysis. The general concept of distortion is to aid in those situations where the expected value of a risk distribution fails to adequately consider the very harsh (but unlikely) consequences included in the risk distribution’s tail. By reallocating density toward the tail using a distortion function, the expected value better reflects the higher severities. The open literature mentions at least eight distortion functions, each having one, two, or three parameters, but there is no specific guidance provided for selecting a distortion function or its parameters. Each distortion function can transform the risk distribution in different ways, depending on the parametric values selected.

### ***1.3 Problem Statement and Research Objectives***

Clearly, questions remain as to what constitutes an appropriate methodology for risk analysis over a wide variety of applications, including reliability and industrial engineering, medicine, finance, and national defense. For instance, which theoretical risk distributions, other than the exponential and Weibull, should be considered? What effects do different distortion functions have on those risk distributions that constitute a versatile set of choices? In those situations where risk analysis involves measuring deficits in current capability, is one distortion function more effective than another when applied to a specific distribution? Can a generalized rule base be established for choosing a proper distribution/distortion combination? Finding answers to these questions aids in bridging some important gaps in the risk analysis literature.

The problem of this research is to propose a methodology for selecting appropriate probability distributions and distortion functions, including distortion parameters, that decision makers can apply in prioritizing risks associated with shortfalls in capabilities. While the primary backdrop here is the USAF's CRRRA process, the contribution of this research is applicable to risk analysis in a wide variety of disciplines.

The thesis has three primary objectives:

1. To propose a concise (but versatile) list of theoretical distributions that can be used to model a wide variety of risk scenarios;
2. To investigate the interactions between four specified parametric distributions and four primary distortion functions included in the literature, based on the manipulation of the distortion parameters over a given set of parameter ranges;
3. To establish a generalized rule base for selecting the "best" combination of distribution and distortion function for a specific risk scenario.

## ***1.4 Assumptions and Limitations***

One of the primary assumptions of this work is that the mathematical expectation is the risk measure of choice among those presented in the thesis; other candidates, for example, would include mean-variance methods and conditional expectation. Another important assumption is that the decision maker trusts the recommendations of his SMEs, which has an effect on the amount of distortion applied to a given distribution.

An important limitation of this thesis is that only four distortion functions and four probability distributions are considered in modelling risks. While virtually no limits are placed on the possible values for the distribution parameters (the lone exception is the case of the Weibull, where the shape parameter is held to 3.5 or less), the distortion parameters cover only a portion of their full ranges. This is done so distortions can be compared over regions in which they are shifting approximately equal amounts of density.

Because expectation is the risk measure used in this thesis, and only four distortion functions are considered, the fact that analytical expressions for the mathematical expectation could only be obtained in a subset of all the cases is also a significant limitation. Additionally, no simple numerical method could be found for computing the expectation of the one multi-parameter distortion under study.

## ***1.5 Outline of the Thesis***

The next chapter introduces background literature on risk and selected probability distributions used in risk analysis, and surveys the current landscape in the theory of distorting transformations and their applications. In Chapter 3, a small-yet-versatile set of four probability (severity) distributions is selected for use in the thesis, and a methodology for assigning one probability distribution to each risk scenario (as based on SME-inputs) is described. Four common distortion functions are

applied to the selected distributions in Chapter 4, their effects within specified parameter ranges analyzed, and some broad-based recommendations on the appropriateness of each distortion in different circumstances recorded; the recommendations come from two primary methods of measurement. Chapter 5 applies the distribution selection methodology and the distortions to a specific notional problem which could be encountered during the CRRA process. Chapter 6 summarizes the thesis and suggests future research based on the scope and limitations of the results.



## 2. Review of the Literature

In a general sense, most people have a basic idea of what is meant by *risk* – one formal definition is “the chance of injury, damage, or loss” [10:1228]. Risk *management* includes many different activities, from identifying to reporting to assessing, analyzing, and handling risks [14]. Risk *assessment* and risk *analysis*, unfortunately, are sometimes interchanged in the literature. According to the International Council on Systems Engineering (INCOSE),

Risk assessment is the process of characterizing or quantifying both the likelihood of occurrence and the severity of the consequences of identified risks. Risk analysis is the process of evaluating alternatives for handling the assessed risks [14].

Thus the retrieval of SME opinions can be classified as a risk assessment venture; the primary focus of this thesis, however, is risk analysis.

### 2.1 *Risk Concepts, Distributions, and Measures*

Risk, and the mathematical approaches to quantifying it, have been the subject of a great number of books and journal articles. Many of these works consider risk in the fields of finance, health, and reliability engineering. While the goal of this chapter is not to review the entire field of risk, some background information on risk concepts, probability distributions, and numerical measures is required.

#### 2.1.1 *General Concepts of Risk*

Haimes [11:19] defines risk as “a measure of the probability and severity of adverse effects.” A somewhat similar definition is offered by Modarres *et al.* [20:18], whose concept of risk includes three parts: the “scenario of events that lead to hazard exposure,” the likelihood or probability of the scenario, and “the consequence (or

evaluation measure) of (the) scenario, e.g., a measure of the degree of damage or loss.” The difference between these definitions is that Modarres et al. [20] include the event scenarios which may (or may not) lead up to a negative outcome. Otherwise, both include the probability of an event occurring and its harshness, given that the event has occurred.

Correspondingly, Woodward [32:3-1] makes a clear but subtle distinction between risk and severity. He defines the severity distribution as a conditional probability, where it is assumed that a negative consequence of some severity will occur with certainty. Similarly, the risk distribution includes all of the severity distribution, but also the possibility of no negative consequence whatsoever (i.e., it is uncertain that *any* event will occur). Hence, when the occurrence of an adverse effect is known with certainty, the risk and severity distributions are equivalent. Haimes [11] is, in effect, describing the severity distribution, while Modarres *et al.* [20] are describing the risk distribution. Note that other names are frequently attached to the severity and/or risk distributions, such as survivor function, reliability function, decumulative distribution function (d.d.f.), and complementary cumulative distribution function (c.c.d.f.).

The severity distribution can be envisioned as a curve which plots each potential severity of a scenario against the probability of experiencing a severity *at least as harsh*, given that some event has occurred (or will occur). Let the non-negative random variable  $X$  be defined as the severity that could be experienced in a scenario (given that an undesirable outcome occurs), and let  $x$  be a realization of that random variable. If  $F(x)$  is the standard cumulative distribution function for  $X$ , then the severity distribution is

$$S(x) = 1 - F(x) = P(X > x). \quad (2.1)$$

The distribution  $S(\cdot)$  satisfies the following properties:

1.  $X$  is continuous on  $\mathbb{R}_+$ .
2.  $S(x) \in [0, 1]$  for all  $x \in \mathbb{R}_+$ .
3.  $S(0) = 1$ .
4.  $\lim_{x \rightarrow \infty} S(x) = 0$ .
5.  $S(x)$  is continuously decreasing; i.e., if  $x, y \in \mathbb{R}_+$ , and  $x < y$ , then  $S(x) > S(y)$ .

As previously mentioned, the risk distribution,  $R(x)$ , is also defined as the probability of experiencing a severity at least as harsh as  $x$ , but without the conditional aspect of an assured event occurrence. The risk distribution is more difficult to work with, since it is a mixed discrete-continuous random variable with a discrete point mass representing the probability of no adverse effect occurring (see Figure 2.1 for an example of a risk distribution). Due to this difficulty, the thesis will work primarily with the severity distribution, although the risk distribution will return to prime importance in Chapter 5.

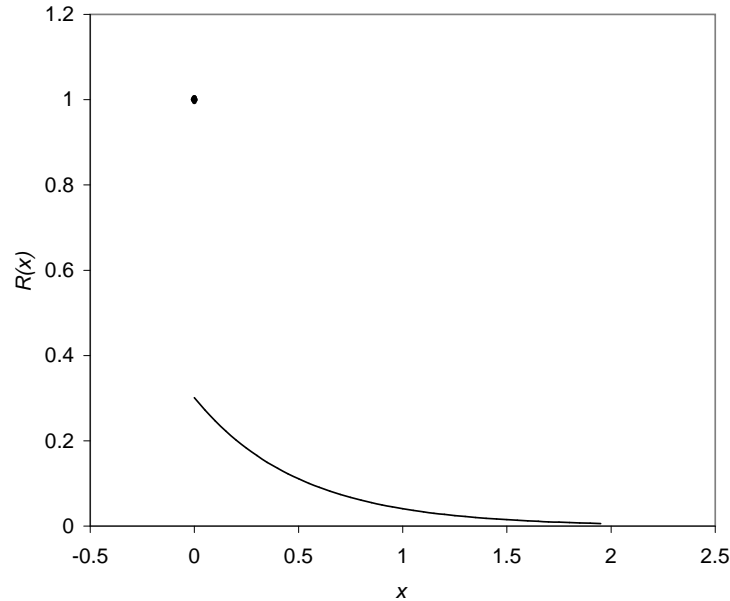


Figure 2.1 A sample risk distribution plot, with  $P(X = 0) = 0.70$  and an exponential tail.

### 2.1.2 Risk Distributions

In the study of risk, many different parametric probability distributions are encountered. While some are identified with specific disciplines, a few of them are seen across virtually all fields of study. (Because the theory behind distortion functions does not support discrete probability distributions, only continuous distributions will be considered.)

In the field of reliability engineering, probability distributions usually describe lifetimes for individual components and complete systems of components, and many textbooks discuss the most common distribution choices. Ebeling [9] uses only four parametric distributions to model failure times: exponential, Weibull, normal, and log-normal. Modarres *et al.* [20] add the gamma distribution to Ebeling's list; the gamma is directly related to the exponential distribution, specifically as the sum of a given number of identically distributed exponential random variables.

The health sciences use risk distributions to model safe/unsafe exposure and dosage amounts, and to make predictions about an individual's predisposition to specific medical conditions. Spread throughout Cox's [6] text are applications of the beta, exponential, Gaussian (normal), Weibull, log-normal, and Pareto distributions. Neely [22], primarily concerned with chemical exposure risk, demonstrates models based on the normal, exponential, and logistic curves. Finally, Hallenbeck's [12] work on occupational health mentions the log-normal, log-logistic, and Weibull distributions.

Financial applications of risk distributions are observed primarily in investment and portfolio theory and actuarial science, where accurate risk assessments are the difference between solvency and bankruptcy. Some key continuous distributions in the actuarial sciences, as per Kaas *et al.* [16], include the uniform, normal, gamma, exponential, beta, log-normal, and Pareto.

Although his text covers primarily reliability engineering, Leemis [18] presents one further consideration than the previously mentioned authors. In discussing different reliability distributions, his goal is not to simply give examples showing how specific distributions can be applied, but rather to discuss the range of modelling capabilities one can achieve using different distributions. Specifically, Leemis [18:94] makes the point that each reliability distribution has specific shapes it is capable of modelling in regards to the hazard rate function.

The hazard rate function, defined as  $h(x) = f(x)/S(x)$  (i.e., the ratio of the probability density to the survivor function), is popular in reliability engineering because it can be interpreted as the expected number of failures per unit time at time  $x$  [18]. In risk applications, the need for the hazard function is not as clear. According to Leemis [18], a “probabilistic interpretation” of the hazard function is

$$h(x)\Delta x = P(x \leq X \leq x + \Delta x \mid X \geq x). \quad (2.2)$$

As applied to risk analysis, this is the conditional probability of being within some specified interval of severity  $[x, x + \Delta x]$ , for small values of  $\Delta x$ . While this definition has meaning, the hazard function is not usually observed in the risk literature (especially that related to finance).

The concept of shape is still important in risk analysis, but more so as it applies to the density function,  $f(x)$ . For a risk scenario, the SME(s) may be able to provide both a range (either finite or infinite) and a mode, the most frequently expected outcome. In the risk case, we would like to have a selection of distributions available that can reflect symmetric densities and various degrees of skewness in both directions.

Leemis [18] also sorts distributions by the number of parameters required to define them. All other things being equal, he implies that a distribution with fewer

parameters is generally preferable to one with more parameters, since having more parameters means that more parameters must be statistically estimated.

As a final observation regarding risk distributions, Hershauer and Nabielsky [13] list nine “situations for knowledge” which reflect an ever-increasing amount of doubt in the potential distribution of a risk. On one end of this continuum, a very similar risk scenario is already established and an accurate distribution, reflecting repeated historical data, has already been assigned to the similar scenario. At the other end, only the slightest amount of knowledge is known about the scenario (extreme uncertainty); this case is one step worse than the SMEs knowing only the range of potential risks. We should therefore select distributions which may reflect this continuum regarding the amount of information the SMEs might actually be able to provide about the risk scenario.

### 2.1.3 Risk Measures

Once a severity (or risk) distribution has been selected, a simple way to characterize the distribution is required so that it can be compared to scenarios presented by other choices. Many methods seek to quantify (summarize) the entire severity distribution in a single risk measure. Some of the most popular risk measures will be reviewed here to set the stage for the principle subject of this thesis, distortion functions.

Due to its widespread use, a logical place to begin the discussion of risk measures is the mathematical expectation or *expected value*, a basic concept found in any elementary probability and statistics text. For a non-negative continuous random variable  $X$  with probability density function (p.d.f.)  $f(x)$  and survivor function  $S(x)$ , the expected value is

$$E(X) = \int_0^{\infty} x f(x) dx = \int_0^{\infty} S(x) dx. \quad (2.3)$$

The problem with expected value as a risk measure is that it serves only as the center of mass for the distribution, meaning it is generally “dampened out” by the values with the greatest relative frequency. For instance, a very unlikely yet catastrophic consequence will likely be more than counterbalanced (even almost totally hidden) by more mild yet highly likely consequences, thus deceiving the decision maker as to what truly constitutes a reasonable risk. The other well-known measures of central tendency, median and mode, also deserve mention. While the mode (the severity with the highest relative frequency) does have specific applications, it is undesirable as a stand-alone risk measure because it fully ignores any low-frequency events reflected in the tails of the distribution. The median (or middle value) of the severity distribution is generally recommended for use when the given distribution is asymmetric to avoid the influence of values which exist in the extended tails of a distribution. Thus, of the three primary measures of central tendency, only the (unconditional) expectation measure is able to weight the tails of the distribution to any extent; it is therefore the most appropriate of the three in these cases, despite its flaws.

A more developed option, then, might also include the variance (or standard deviation) of the distribution. Using the variance or standard deviation in isolation has the obvious flaw (in general) of not revealing the expectation of the risk, even though most people would agree that smaller variance around a risk should reduce risk overall because the outcome is “more certain.” A better method might simply use the variance as a “tie-breaker” among those severity distributions with equal or nearly equal expected severities [16:223], but this is a relatively unlikely case and the definition of “nearly equal” is situation-dependent. (When both the mean and variance of the risk distribution are equal, the third moment, or “skewness,” of the distribution is sometimes used as another tie-breaker [7].) As a more sophisticated mean-variance methodology, Sarin and Weber [23] discuss an entire class of “risk-value models,” the most basic of which mathematically *combine* variance (the measure of risk) with expected value (the measure of value) as a single risk measure.

Many such models from the financial arena incorporate the use of economic utility functions, which reflect the value of “the next dollar” to the subject’s risk analysis; as one might expect, more advanced risk measures frequently require such decision maker input.

Conditional expectation is a risk measure entirely devoted to considering the events in the tail of a distribution, with the definition of “tail” left open to the analyst. One conditional risk measure that appears prominently in the financial literature is Value at Risk (VaR). VaR is not a classic mathematical expectation; rather, the analyst chooses a value,  $\alpha \in (0, 1)$ , corresponding to the certainty he or she wishes to have over his or her ability to cover a potential loss, much like  $\alpha$  in statistical hypothesis testing. On a scale of net worth [24], if the  $(100 \times \alpha)\%$  point in the distribution is to the right of zero, then VaR reflects a “safe” position, while to the left of zero further assets must be added to consider the position safe. Some of the main advantages of VaR include its ease of calculation and comprehension; a significant disadvantage will be discussed shortly.

True conditional expectation, sometimes called “conditional VaR” (CVaR), is very similar to VaR, but instead of simply marking the risk distribution at the  $(100 \times \alpha)\%$  point, the mathematical expected value of the entire tail “isolated” beyond the  $(100 \times \alpha)\%$  mark is considered. This measure, then, is actually more conservative than VaR.

Mathematical expectation of parametrically distorted survivor functions is the last of the risk measures to be discussed. Due to their prominence in the thesis, distortion functions will be addressed more completely in the following section.

## ***2.2 Parametric Distortion Functions***

As a specific mathematical approach to measuring financial risk, there is a relatively small amount of work in parametric distortion functions (although this is



an active area of research). Most papers either describe the mathematics behind the science, show specific applications, or compare distortion to other financial risk measures. The focus of this section is to present the rudimentary concepts of distortion functions, with a slight emphasis on literature that may aid in selecting a distortion function or specific distortion parameters.

A function  $g$  is a distortion function if it satisfies the following [31:338]:

1.  $g : [0, 1] \rightarrow [0, 1]$  is an increasing function,
2.  $g(0) = 0$ , and
3.  $\lim_{u \rightarrow 1} g(u) = 1$ .

Since any function  $g$  which meets the above criteria is a distortion function, there are theoretically an infinite number of them. However, Wang [30] specifically works with only a few, and all are directly related to a single transform known as the gamma-beta distortion. The most extensive listing of distortion functions in the literature comes from McLeish and Reesor [19], who formally define eight specific transformations: gamma-beta, beta, proportional hazard, dual power, gamma, exponential, normal, and Esscher.

Since the distributions under consideration in the thesis are assumed to be parametric functions themselves, the combination of a distortion function  $g$  and its subject c.c.d.f.  $S(x)$  is a composition of functions, also resulting in a survivor function denoted by

$$g(S(x)) \equiv (g \circ S)(x). \quad (2.4)$$

A distortion function (transformation) pushes risk density toward the tail of the probability distribution under consideration. In this way, rather than “cutting off” a portion of the distribution to emphasize the tail (as per conditional expectation), the entire distribution is re-shaped so the standard mathematical expectation can still be employed. A wide variety of re-shaping effects and degrees of effect are possible.

### 2.2.1 Coherency

In the literature, distortion is closely tied to the concept of “coherency,” outlined by Artzner *et al.* in [2] and further developed in [3]. The impetus behind coherency stems from the popularity (in the financial fields) of the VaR risk measurement (previously described). Artzner *et al.* [2], in essence, noticed that in calculating VaR, if a firm makes  $k$  investments that are below the  $(100 \times \alpha)\%$  risk threshold, the total amount invested is still subject to Bonferroni’s inequality, i.e., the overall risk involved is greater than the firm’s stated risk limit. In this situation, a large firm would be required to always have a tremendous amount of cash on hand to cover potential losses, while  $k$  individual firms, each holding one of the investments, would not be required to hold any cash at all. From this scenario, Artzner *et al.* [2:69] establish four relations that a *coherent* risk measure must possess, with coherent meaning that the risk measure accurately portrays the way financial markets operate. (In the following statements,  $\rho$  is a risk measure,  $X$  and  $Y$  are non-negative random variables of risk severity,  $n$  is a number, and  $t$  is a positive number.)

- i. Sub-additivity.  $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- ii. Homogeneity.  $\rho(t \cdot X) = t \cdot \rho(X)$
- iii. Monotonicity.  $\rho(X) \geq \rho(Y)$ , if  $X \leq Y$
- iv. Risk-free condition.  $\rho(X + r \cdot n) = \rho(X) - n$

The four properties listed above establish necessary properties for measuring risk [4:576]. Property (i), as alluded to in the VaR discussion, assures that the sum of the individual risk measures serves as an upper bound for the risk measure of the sum [2:69]. Convexity of the risk measure comes from the combination of (i) and (ii). Property (iv) allows a financial institution to reduce risk by investing  $n$  dollars at  $r\%$ , risk-free. Property (iii) is a direct result of the characteristics of the c.c.d.f. Once established that a risk measure  $\rho_g$  is coherent, McLeish and Reesor [19:138]

prove that the associated distortion function  $g$  is concave – in fact, either one implies the other (mathematically, **concavity**  $\Leftrightarrow$  **coherency**).

### 2.2.2 Gamma-Beta Family of Distortions

The distortions that are most frequently mentioned in the literature are the gamma-beta distortion and its close relatives, all of which are included in McLeish and Reesor [19]. The gamma-beta distortion is defined as

$$g_{GB}(u) = \int_0^u K t^{a-1} (1-t)^{b-1} \exp(-t/c) dt, \quad (2.5)$$

where

$$K^{-1} = \int_0^1 t^{a-1} (1-t)^{b-1} \exp(-t/c) dt \quad (2.6)$$

and

$$u \equiv S(x). \quad (2.7)$$

The gamma-beta does not appear to be used frequently in practice; rather it serves as a baseline for distortions with fewer parameters.

Of the six listed here, the gamma-beta is the only three-parameter distortion function having parameters  $a$ ,  $b$ , and  $c$ . A very important item to note is that *any non-negative values* may be selected for  $a$ ,  $b$ , and  $c$ , but McLeish and Reesor [19] show that using ranges of  $0 \leq a \leq 1$ ,  $b \geq 1$ , and  $c \geq 0$  are sufficient to ensure concavity – and *coherency* – of the distorted risk measure.

The beta distortion,

$$g_\beta(u) = \int_0^u \frac{1}{\beta(a,b)} t^{a-1} (1-t)^{b-1} dt \quad (2.8)$$

where

$$K^{-1} = \beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \quad (2.9)$$

is the gamma-beta distortion when  $c \rightarrow \infty$ . With  $c$  held constant, the beta is a two-parameter distribution, and Wirch and Hardy [31:342] note that the parameters  $a$  and  $b$  can be used to manipulate both the initial gradient and the convergence rate of the survivor function.

The proportional hazards (PH) distortion is widely used in the literature, and is derived from the beta distortion (and naturally the gamma-beta) by holding  $b = 1$  and allowing  $c \rightarrow \infty$ . Thus

$$g_{PH}(u) = u^a, \quad (2.10)$$

and the result is a very simple distortion to work with in practice. While the similarity is not obvious, a multivariate hazard model of the same name is used frequently in the biological sciences.

The dual power (DP) distortion is the other distortion which stems from the beta. In this case,  $a = 1$  and  $c \rightarrow \infty$ , and with the single parameter  $b = \kappa$  free to vary the resulting distortion is

$$g_{DP}(u) = 1 - (1 - u)^\kappa. \quad (2.11)$$

Within the literature, this distortion has the most clear interpretation: Wirch and Hardy [31:340] state that for an integer value of  $\kappa$ , the risk measure equates to “the expected value of the maximum of a sample of  $\kappa$  observations of  $X$ .”

The gamma distortion, defined as

$$g_\gamma(u) = \int_0^u K t^{a-1} \exp(-t/c) dt \quad (2.12)$$

where

$$K^{-1} = \int_0^1 t^{a-1} \exp(-t/c) dt, \quad (2.13)$$

is another two-parameter transformation. In this case the fixed value is  $b = 1$ , allowing  $a$  and  $c$  to be manipulated.

Finally, the exponential (EX) distortion also has but a single parameter:

$$g_{EX}(u) = \frac{1 - e^{-u/c}}{1 - e^{-1/c}}. \quad (2.14)$$

This distortion is the recognizable exponential CDF restricted to  $[0, 1]$ .

There is almost no published literature available regarding the selection of a distortion function or its associated parameter values for a given set of circumstances. Other than their specific interpretation of  $\kappa$ , Wirth and Hardy [31:342] come closest, making two general observations regarding transformation parameters. First, they associate the parameters with a decision maker's risk aversion level toward risk in the far right tail of the distribution. Second, they state an opinion that the selection of distortion parameters is mostly a "political" decision [31:347].

### **2.3 Summary**

The chapter has defined general concepts of risk and differentiated between risk and severity. Various disciplines were mentioned in which risk plays a prominent role, and the common distributions observed in those disciplines were listed as a prelude to Chapter 3. Afterwards, some popular risk measures were examined, including expectation, mean-variance models, conditional expectation (including VaR from financial circles), and finally parametric distortion functions. Due to the leading role they play in the thesis, distortions were subsequently examined in further detail, and the apparent lack of information regarding distortion function and parameter selection noted. Coherency was also discussed, especially in terms of how it affects the selection of distortion parameters.

### 3. Assigning Distributions to Risk Scenarios

The primary objectives of this thesis are to propose a concise list of theoretical distributions that can be used to model a wide variety of risk scenarios; conduct a formal investigation of the effects of distortion functions on probability distributions (based on the manipulation of the distortion parameters); and establish generalized guidelines for selecting specific combinations of distortion and distribution. This chapter will address the first of those objectives by discussing the criteria used in selecting a set of four distributions suitable for risk analysis and then providing a simple method an analyst might employ in assigning one of those distributions (and specific parameter values) to SME-provided data.

#### *3.1 Selection of Severity Distributions*

Some basic considerations can aid in selecting a set of distributions to use in modelling risk scenarios. In addition to those criteria provided in Chapter 2, it should be recognized that a risk scenario may or may not cover the full range of potential severity, and thus both bounded (finite) and unbounded (infinite) distributions should be included. Combining these criteria, the selected distributions should:

1. represent useful distributions from a variety of risk disciplines, including reliability engineering (RE), health sciences (HS), and finance and insurance (FI);
2. cover a range of p.d.f./c.c.d.f. shapes;
3. cover both finite and infinite risk/severity ranges of the random variable; and
4. allow for parameter choices that correspond to information that is likely to be available from an SME, while simultaneously limiting the number of parameters that must be estimated.

Table 3.1 summarizes the distributions initially considered as candidates for the parametric analysis; all of the distributions mentioned in Chapter 2 are included here, and a few additional ones have been added. The first column lists the distribution name and parameters [15], the second column the corresponding p.d.f., and the third the disciplines where the distribution is most frequently observed (as per the abbreviations in consideration 1).

While the table presents a set of distributions to consider, the problem with including all of the listed distributions is simply one of combinatorics. If all 11 of the distributions and all six of the gamma-beta distortion functions were examined, there would be 66 possible combinations even before varying the individual distortion parameters. When including distortion parameter levels, if three levels were chosen for each, we would have in effect 54 different distortions (using the number of parameters for each of the gamma-beta family), and  $11 \times 54 = 594$  individual combinations. Thus, the number of distributions under examination was held to four, including one single-parameter distribution, two dual-parameter distributions, and one three-parameter distribution. The selected distributions satisfy the criteria set forth by considerations 1 through 4 (particularly in terms of p.d.f. shapes and ranges) and are now described in greater detail.

Table 3.1 Common continuous distributions.

Distribution (parameters)	Probability Density Fct. (p.d.f.)	Disciplines
Beta ( $\alpha, \beta$ )	$f_X(x) = \left[ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \right] x^{\alpha-1}(1-x)^{\beta-1}$	HS, FI
Exponential ( $\lambda$ )	$f_X(x) = \lambda e^{-\lambda x}$	RE, HS, FI
Gamma ( $\lambda, \kappa$ )	$f_X(x) = \frac{\lambda(\lambda x)^{\kappa-1} e^{-\lambda x}}{\Gamma(\kappa)}$	RE, FI
Logistic ( $m, b$ )	$f_X(x) = \frac{e^{-(x-m)/b}}{b[1+e^{-(x-m)/b}]^2}$	HS
Log-logistic ( $\lambda, \kappa$ )	$f_X(x) = \frac{\lambda \kappa (\lambda x)^{\kappa-1}}{[1+(\lambda x)^\kappa]^2}$	HS
Log-normal ( $\mu, \sigma$ )	$f_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}$	RE, HS, FI
Normal ( $\mu, \sigma$ )	$f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$	RE, HS, FI
Pareto ( $\lambda, \kappa$ )	$f_X(x) = \frac{\kappa \lambda^\kappa}{x^{\kappa+1}}$	HS, FI
Triangular ( $\theta_1, \theta_2, m$ )	$f_X(x) = \begin{cases} \frac{2(x-\theta_1)}{(\theta_2-\theta_1)(m-\theta_1)} & \text{if } \theta_1 \leq x \leq m \\ \frac{2(\theta_2-x)}{(\theta_2-\theta_1)(\theta_2-m)} & \text{if } m < x \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$	FI
Uniform ( $\theta_1, \theta_2$ )	$f_X(x) = \frac{1}{\theta_2-\theta_1}$	FI
Weibull ( $\beta, \theta$ )	$f_X(x) = \frac{\beta}{\theta} \left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}$	RE, HS



The exponential distribution is a continuously decreasing density function characterized by a single parameter,  $\lambda$ . This distribution is generally used in lifetime modelling applications, especially involving electronic components, but is one of the most commonly encountered distributions across all disciplines. The exponential is also the only continuous distribution with the memoryless property, and the only single-parameter distribution among those mentioned in the table. The single parameter  $\lambda$  can be used to directly compute both the mean ( $\mu = 1/\lambda$ ) and variance ( $\sigma^2 = 1/\lambda^2$ ) of the distribution. If the SME only knows that *some* severity will occur and can provide an estimated mean severity, the exponential is likely the modelling distribution of choice. As such a core distribution to the field of probability, the exponential is a logical selection as one of the four distributions for the study. See Figure 3.1 for an example of an exponential density function.

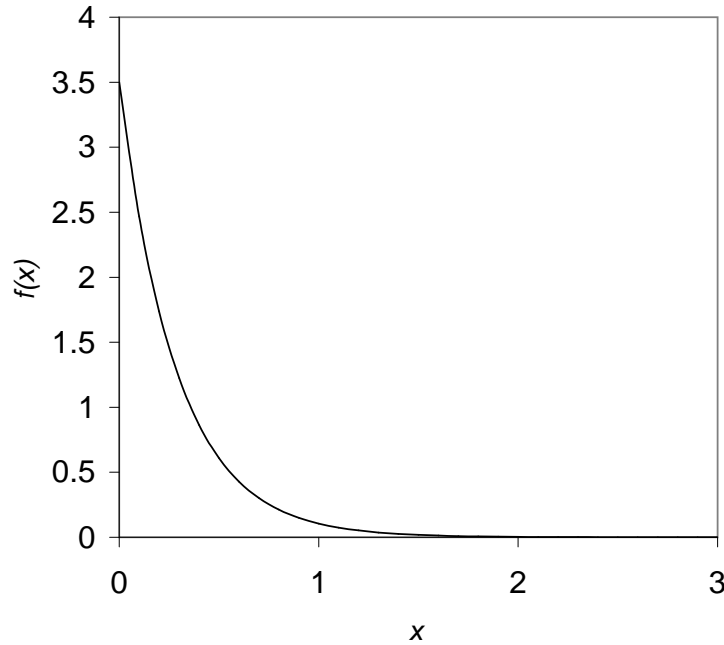


Figure 3.1 Example exponential probability density function (p.d.f.),  $\lambda = 3.5$ .

The Weibull distribution can take a variety of shapes, from exponential to approximately normal, and thus can be used to model a range of risk/severity distribution shapes. The Weibull displays this versatility through two parameters,  $\beta$

and  $\theta$ . The  $\beta$  parameter is typically called the shape parameter, since it determines whether the distribution more closely resembles an exponential ( $\beta = 1$ ) or (possibly right-skewed) normal distribution ( $1 < \beta \leq 3.5$ );  $\theta$  is the scale parameter, which influences both the distribution's mean and dispersion. Like the exponential, the Weibull is encountered in many fields (including the health sciences), but perhaps most frequently in reliability engineering, where it sometimes models lifetimes of mechanical components. As Johnson *et al.* [15] state that the Weibull “is undeniably the (continuous) distribution that has received maximum attention during the past (32) years,” it should be included for study. See Figure 3.2 for an example of a Weibull density function with specific shape and scale parameters.

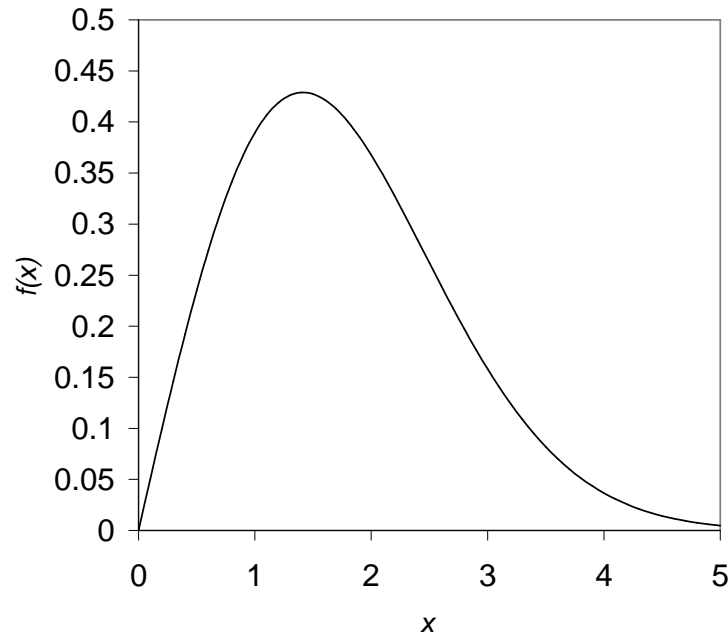


Figure 3.2 Example Weibull probability density function (p.d.f.),  $\beta = 2$ ,  $\theta = 2$ .

The uniform distribution is most frequently observed in the generation of random numbers, but it also has applications in finance and insurance, life testing, and traffic flow problems, among others [15]. The uniform is another two-parameter distribution, where  $\theta_1$  and  $\theta_2$  define the range of possible values for the uniformly-defined random variable to assume. Within the defined range, all values have equal

likelihood of occurrence. The Weibull and exponential distributions differ from the uniform in that the random variables they describe can take on any value  $x \geq 0$ , whereas the uniform sets limits on both the upper *and* lower values the random variable can assume,  $\theta_1 \leq x \leq \theta_2$ . Thus the uniform distribution is appropriate for modelling risks/severities where definite upper and lower bounds on severity are known, and especially in the case where no other information except the bounds is available. For these reasons, the uniform is a reasonable selection. See Figure 3.3 for an example of a uniform density function.

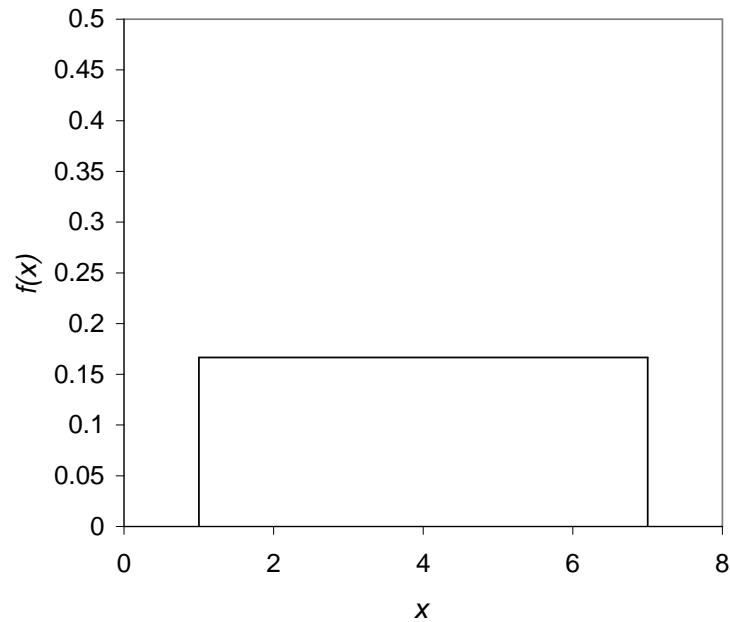


Figure 3.3 Example uniform probability density function (p.d.f.),  $\theta_1 = 1$ ,  $\theta_2 = 7$ .

Finally, the triangular distribution offers many of the same advantages of the uniform and is used in a similar set of disciplines, but includes additional information. The triangular distribution has parameters  $\theta_1$ ,  $\theta_2$ , and  $m$ , where  $\theta_1$  and  $\theta_2$  are defined as in the uniform distribution, and  $m$  is the most frequently observed occurrence ( $\theta_1 \leq m \leq \theta_2$ ), or the mode. Like the uniform, the triangular is easy to use and explain; it presents a useful option in risk analysis and will be the final distribution selected. See Figure 3.4 for an example of a triangular probability density function.

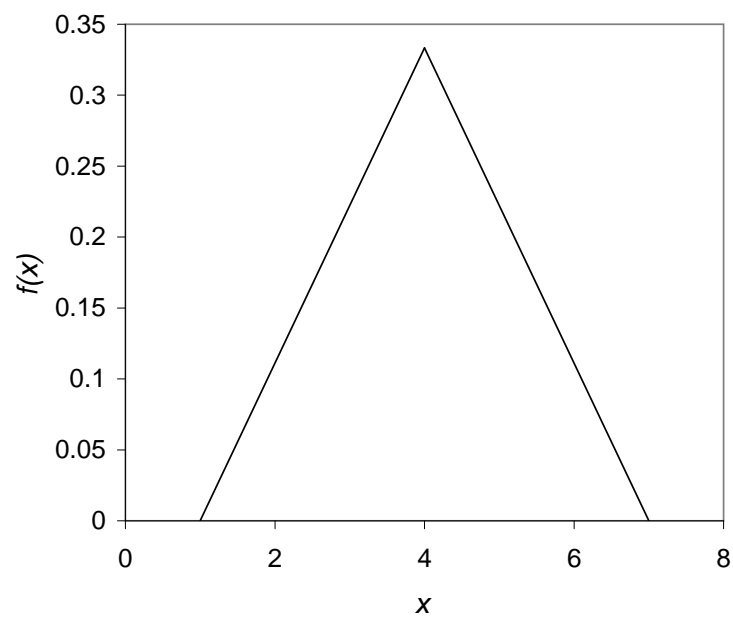


Figure 3.4 Example triangular probability density function (p.d.f.),  $\theta_1 = 1$ ,  $\theta_2 = 7$ ,  $m = 4$ .

### 3.2 Risk Scales

With a reasonable set of severity distribution choices selected, the analyst's next goal is to assign one of them to SME-provided data and specify its distribution parameters. Before that task can begin, however, a brief discussion of risk scales is required. The risk scale provides a context by which the SMEs can convey their opinions.

A natural scale people use in rankings is “an integer from 1 to 10.” This, or any other *linear* scale, can serve as a starting point for discussion. One problem with “1 to 10” as a severity scale is the implication that, for instance, two events which rate a 5 might be worth a single one which rates a 10, which may or may not be true. This problem can be overcome by mapping the “1 to 10” score to another scale using a function. If  $x$  represents the “1 to 10” score and  $y$  is the space of “correct” severity weighting, some typical functions that might be used to map  $x \rightarrow y$  include  $y = x^2$  or  $y = \sqrt{x}$ , depending on how the linear inaccuracy of the scale should be corrected. A second problem is that the discrete scale does not allow a continuous score. By releasing this restriction to allow, for example, any continuous score in the interval  $[0,10]$ , the ability to draw small distinctions between severities is improved. Of course, the concept of “limiting” the greatest consequence to a ranking of 10 is artificial; one can allow risks to be scored on a  $[0, \infty)$  scale while only describing risks within the context of  $[0,10]$ . In this fashion, if some negative outcome needs to be called a “14.7,” the freedom exists to do so.

For the remainder of this chapter and the next, a continuous version of the CRRRA risk scale will be used as a basis for examples. This scale is fully enumerated in Table 3.2.

Table 3.2 Continuous version of CRRA severity scale.

Severity Factor	Minor	Modest	Substantial	Major	Extensive	Catastrophic
CRRA Range	$[0, 1]$	$(1, 2]$	$(2, 3]$	$(3, 4]$	$(4, 5]$	$(5, 6+)$

### 3.3 *Fitting Severity Distributions to Expert Data*

With a risk context in-hand, assigning a severity distribution to a specific risk scenario requires several considerations from the SMEs. First, we would like to know the severity limits on the consequences related to the risk scenario. In many instances, any severity from *minor* to *catastrophic* might be observed, i.e., on the range  $[0, \infty)$ . Conversely, however, the SMEs might believe that only a segment of the entire severity range could be realized, say  $[0.5, 3]$  on the continuous scale.

Second, within the established limits of severity that might be observed, we would like to know the most likely categorization of severity that might be experienced because of this vulnerability (i.e., the mode). This is critical because knowing the mode affects the overall symmetry or skewness of the distribution, and not all distributions can reflect both types of skew. As a relevant example, the Weibull distribution can reflect a mode of zero if  $\beta = 1$ , and as  $\beta$  is increased the density becomes approximately symmetric. However, the Weibull could not be used to reflect a density where the mode is “toward” the right, i.e., decidedly left-skewed.

Third, we would like to know if the mean of the distribution is available. In deciding to produce this value, the SMEs must keep in mind that in a left-skewed distribution, the mean will sit to the left of the mode, and vice versa for a right-skewed distribution.

Using the selected four distributions, then, a simple method can be devised for determining which distribution to assign to each risk scenario to model its (potential)

severity. The responsible SME (or team) for each vulnerability can be asked three specific questions:

1. What is the *range* of severities that might be encountered in this risk scenario?
2. Within the specified range, what is the *most likely* severity or categorization of severity that might be experienced (i.e., the mode)?
3. Can an expected value be specified?

Where numerical answers are provided, SMEs should be encouraged to provide accurate, decimal responses whenever possible.

These three questions probe for ever-increasing amounts of detail, and only the first question *must* be answered (since a uniform distribution can be assigned at that point). The other questions only increase the analyst's ability to narrow down a more "detailed" distribution. Once the first question, at the least, has been answered, the decision tree shown in Figure 3.5 may be used to select one of the four distributions under discussion.





As an example, suppose that the SME or expert team provides the following information: the range of possible categories includes *substantial* to *extensive*, and the most likely outcome is *extensive*. No information about the mean is provided, and no numerical data can be agreed upon among the SMEs. Entering the tree, since the category range does not include *minor*, the exponential and Weibull distributions are eliminated because they do contain zero, or *minor*, severity. Since the mode is known, a triangular distribution can be fit rather than a uniform.

### 3.4 *Determining the Distribution Parameters*

Once the distribution is decided upon, its parameters still need to be established. The discussion of distribution parameter choices will proceed from the easier cases (uniform and triangular) to the only slightly more difficult (exponential and Weibull) in the following discussion.

#### 3.4.1 *Uniform and Triangular Parameter Selections*

The SMEs can provide either categorical or numerical data in the case of the uniform and triangular distributions. If numerical data is provided, then that data may be used directly. If the SMEs provide categorical data, such as a range of *modest* to *substantial*, then the numerical edges of the associated severity “bins” can be used as the distribution parameters. In this case, the bottom of the *modest* scale is 1, and the top of the *substantial* scale is 3, so if a uniform distribution is fit to the SME inputs, the random variable  $X$  would be distributed uniformly on (1,3).

Now suppose the mode is also provided and is also predicted to be *substantial*. Because no information is provided about where in the *substantial* range the mode will fall, the midpoint of the range is the most reasonable choice. Thus if a triangular distribution can be fit, the distribution is  $\text{tria}(1,3,2.5)$ .

### 3.4.2 Exponential and Weibull Parameter Selections

The calculations for the “unbounded” exponential and Weibull distributions are only slightly more involved than the bounded. To begin with the exponential, we know already that the mode is zero, so attention falls on the mean. To fit an exponential distribution, information about the mean is essential, because the exponential is a one-parameter distribution and that one parameter is the reciprocal of the mean. If only categorical information about the mean is provided, then, like the uniform and triangular distributions, the midpoint of the category can be substituted.

The Weibull distribution is the most difficult case, since two parameters must be specified,  $\beta$  and  $\theta$ . Suppose first that the mode is provided only in categorical terms, and let us assume, for example, that the category is *modest* (recall that *minor* (above nil) through *substantial* categorizations of the mode result in the selection of the Weibull). On the continuous version of the CRRA scale, we might call the mode 1.5. How can both of the Weibull parameters be estimated from this single piece of information? In various references (such as [9]), the shape parameter  $\beta$  is shown to make the Weibull represent a range of shapes from exponential ( $\beta = 1$ ) to essentially normal ( $\beta = 3.5$ ). Beside this range of  $[1, 3.5]$ , we can align the left-half of the continuous CRRA scale (see Figure 3.6) and, *assuming linear rates of change* over each scale, arrive at a  $\beta$  value of 2.25 for this example.

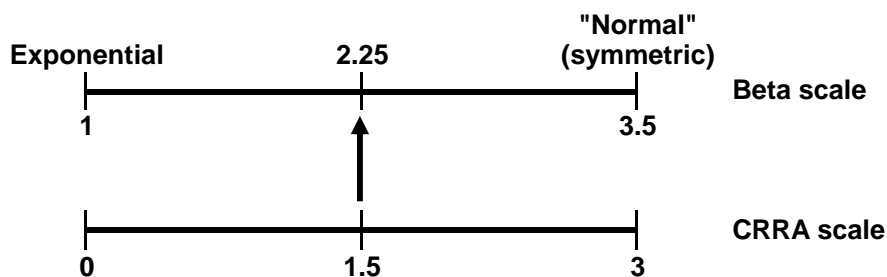


Figure 3.6 Mapping CRRA mode to  $\beta$  parameter scale.

Now armed with  $\beta$  and the mode of the distribution, [17:303] states that the mode of the Weibull distribution is

$$\text{mode}_{Weibull} = \begin{cases} \theta \left( \frac{\beta-1}{\beta} \right)^{1/\beta} & \text{if } \beta \geq 1 \\ 0 & \text{if } \beta < 1. \end{cases} \quad (3.1)$$

The correct value of  $\theta$  is obtained by using the  $\beta \geq 1$  expression for the mode. In other words, set the expression equal to the SME-provided value for the mode, substitute the known value of  $\beta$  into the equation, and solve for  $\theta$ .

Continuing from the previous example, suppose that  $\beta = 2.25$  and the mode is 1.5. Then we would have

$$\theta \left( \frac{2.25 - 1}{2.25} \right)^{1/2.25} = 1.5 \quad \Rightarrow \quad \theta \approx 1.9478.$$

### 3.5 Summary

This chapter has suggested four specific criteria a small-yet-versatile set of severity distributions should satisfy. Four suitable probability distributions were then subjectively selected in line with those criteria, namely the exponential, Weibull, triangular, and uniform. Some considerations regarding what constitutes an effective risk scale followed, establishing a methodology an analyst can use to assign one of the four selected distributions to SME inputs. The chapter closed with a discussion on how distribution parameters can be determined from relatively sketchy SME “consequence” data. Given the vast number of considerations that must go into any risk scenario assessment, sketchy SME data may be all that can be reasonably expected.

## 4. Parametric Analysis of Distortion Functions

In Chapter 3, we established a “small-yet-versatile” set of severity distributions that may be used for risk analysis. The goal of this chapter is to investigate how four members of the gamma-beta family of distortion functions can affect those probability distributions (based on the manipulation of the distortion parameters) and establish generalized guidelines for selecting specific combinations of distortion and distribution.

### 4.1 *Selection of Distortion Functions*

To proceed with distortion function analysis, four of the six distortion functions from the gamma-beta family were selected to combine with the four probability distributions of Chapter 3. In the literature, the proportional hazard, dual power, and beta distortions are frequently examined. However, those three distortions utilize the  $a$  and  $b$  parameters only; to better observe the effect of the  $c$  parameter, the beta was excluded among those three in favor of the exponential distortion (which uses  $c$  alone). (Selecting the three, single-parameter distributions is also advantageous in that expectations can be numerically computed in all cases by using non-integral forms of the distortions – more will follow on this point later.) While the gamma-beta distortion is not usually mentioned in the literature in terms of practical application, it does combine the effects of all three parameters at once. Thus the final distortion function selections are the proportional hazard, dual power, exponential, and gamma-beta distortions.

As a brief review of all four of the selected distortions, the effect of distorting the survivor function of the severity distribution is described in Table 4.1.

Table 4.1 General distortion effects.

Distortion	Parameter	$(g \circ S)(x)$
Proportional Hazard ( $g_{PH}$ )	$0 < a \leq 1$	$S^a(x)$
Dual Power ( $g_{DP}$ )	$b \geq 1$	$1 - (1 - S(x))^b$
Exponential ( $g_{EX}$ )	$0 < c < \infty$	$\frac{1 - \exp(-S(x)/c)}{1 - \exp(-1/c)}$
Gamma-Beta ( $g_{GB}$ )	$a, b, c$ (as above)	$\frac{\int_0^{S(x)} t^{a-1} (1-t)^{b-1} e^{-t/c} dt}{\int_0^1 t^{a-1} (1-t)^{b-1} e^{-t/c} dt}$

#### 4.2 Selection of Distribution Parameters for Experimentation

With four distributions, four distortion functions, and as many as three parameters being varied in the case of the gamma-beta distortion, the parameters of the selected probability distributions were held constant. While thus allowing the focus to fall on the impact of the distortion parameters, the potential drawback was that any interactions between the distortion parameters and the distribution parameters were not considered. The exponential distribution's single parameter was arbitrarily set to 3.5. Since the exponential was part of the study, the Weibull distribution served the experiment best by not mimicking the exponential, i.e., by *not* setting the shape parameter  $\beta = 1$ ; some typical parameter values from [9] were  $\beta = 2$  and  $\theta = 2$  (which is actually a Rayleigh distribution, a specific case of the Weibull). The uniform distribution's two parameters were arbitrarily set to  $\theta_1 = 1$  and  $\theta_2 = 7$ . Since the triangular has upper and lower limits like the uniform, the same limits were used, and the mode of  $m = 4$  was arbitrarily selected to reflect a symmetric triangular distribution.

### 4.3 Selection of Distortion Parameters for Experimentation

Wherever possible, the resulting change in mathematical expectation was the desired measure of distortion. In some cases the expectations could not be computed explicitly and numerical methods were therefore employed. However, in the case of the gamma-beta distortion, the expectation was incalculable even by numerical methods, so the establishment of a different standard by which to measure the effects of distortion was required. This new measure proved useful outside the gamma-beta distortion context, however, and influenced the choice of distortion parameters.

The selected performance measure uses the median of the distribution, specifically the point at which the undistorted distribution is partitioned with equal density on either side. Distortion was then applied, and the amount of density flow from the left side of the *undistorted* median value to the right was measured; the formula used for this measure is

$$R_g = \frac{(g \circ S)(\psi)}{S(\psi)}, \quad (4.1)$$

where

$$\psi \equiv \inf\{x : S(x) = 0.5\}, \quad (4.2)$$

$S(\cdot)$  is the survivor function, and  $(g \circ S)(\cdot)$  is the distorted survivor function. Recalling that all of the distortion functions have the effect of shifting density toward the right, then  $1 \leq R_g \leq 2$ , since by this ratio measurement *all* of the density on the left of the median could theoretically be shifted to the right of the median, but no matter how far to the right the displaced density is pushed the ratio only reflects that it has been pushed *beyond* the fixed undistorted median point. If  $R_g = 1$ , then no distortion is present. Table 4.2 shows the undistorted medians for the selected distributions and their respective parameters.

Because of this “region of sensitivity” of the  $R_g$  measure, the values for the distortion parameters were selected with care. The initial selection of parameters, based solely on what *appeared* to be reasonable values, proved to be too rich; many of

Table 4.2 Undistorted medians for selected distributions.

Distribution	Parameter(s)	Selected Value(s)	Median
Exponential	$\lambda$	3.5	0.198042
Weibull	$\beta$	2	1.666 $\bar{6}$
	$\theta$	2	
Triangular	$\theta_1$	1	4
	$m$	4	
	$\theta_2$	7	
Uniform	$\theta_1$	1	4
	$\theta_2$	7	

the  $R_g$  values observed were close to two, meaning the region of sensitivity where the effects of changes in the distortion parameters could be observed had been exceeded. Compounding that problem, a  $3^k$ -factorial design had been envisioned to study the effects of each parameter ( $a$ ,  $b$ , and  $c$ ) in the gamma-beta distortion, and within the factorial design each of the involved parameters had to have essentially equal power over the  $R_g$  measure so that the interaction effects could be analyzed in a “fair” manner. Finally, due to the face-centered cube preferred for the experimental design, the three values of each parameter had to all be equally spaced. After lengthy experimentation, Table 4.3 shows the distortion parameter values that were selected. (In reviewing the table, recall that  $a = 1$ ,  $b = 1$ , and  $c \rightarrow \infty$  result in no distortion

being applied, and note that distortion *increases* as  $a$  and  $c$  are *decreased*, while the opposite is true for  $b$ .)

Table 4.3 Selected distortion parameter treatments.

Distortion (Parameter)	Selected Value(s)		$R_g$ (% density shift)
Proportional Hazard ( $a$ )	High	0.9	1.07 (7%)
	Mid	0.75	1.19 (19%)
	Low	0.6	1.32 (32%)
Dual Power ( $b$ )	Low	1.1	1.07 (7%)
	Mid	1.3	1.19 (19%)
	High	1.5	1.29 (29%)
Exponential ( $c$ )	High	3.6	1.07 (7%)
	Mid	2.2	1.11 (11%)
	Low	0.8	1.30 (30%)

#### 4.4 Response Surface Analysis of Gamma-Beta Distortion

A face-centered cube (FCC) was the design choice for response surface analysis of the gamma-beta distortion. With three levels selected for each of the three parameters ( $a$ ,  $b$ , and  $c$ ) of the gamma-beta, a total of  $3^3 = 27$  design points were included in the experiment, with the limits taken from Table 4.3. Although the responses are entirely deterministic, the software tool (Design Expert 6.0) allowed for the use of



five additional center runs, and these were included in the experiment for a total of 32 observations. (Note that in the FCC, however, no additional benefit is derived after the second center run [21].)

In employing a second-order model including the first-order interactions (i.e.,  $A$ ,  $B$ ,  $C$ ,  $A^2$ ,  $B^2$ ,  $C^2$ ,  $AB$ ,  $AC$ , and  $BC$ , where each distortion parameter is represented by its corresponding capital letter), the results were nearly identical across all four of the distributions included for analysis (see Figures 4.1 through 4.4). The  $A^2$  term was the only insignificant effect in every case, with  $A$ ,  $B$ ,  $C$ , and  $C^2$  accounting for about 99% of the error sum of squares throughout. There were no diagnostics required for any of the four models because none of the basic assumptions of residual normality, constant error variance, or independence between observations appeared to have been violated. The means used to test the assumptions included the normal probability plots of residuals, residuals versus predicted plots, residuals versus run order plots, residuals versus individual factors plots, Cook's distances, and predicted versus actuals plots.

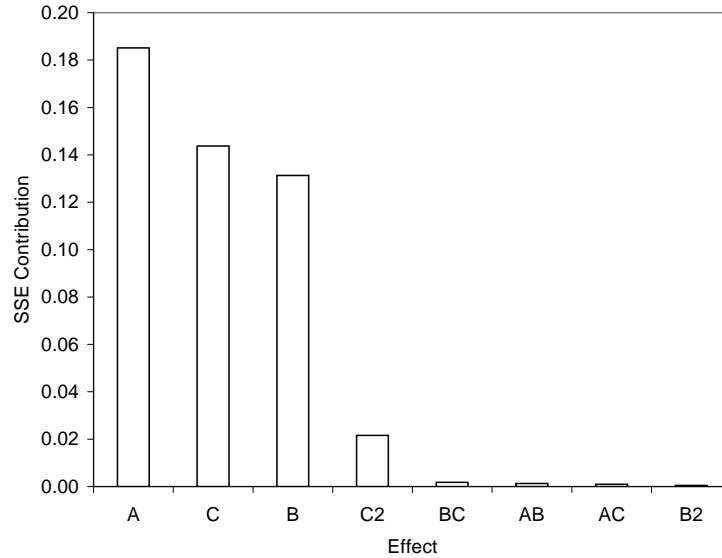


Figure 4.1 Ordered sums of squares associated with estimated effects of  $R_g$  for gamma-beta distortion, exponential(3.5) distribution.

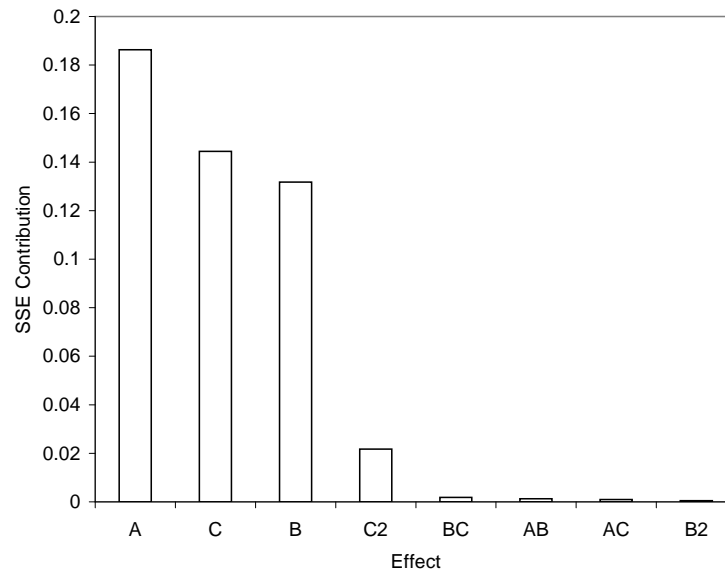


Figure 4.2 Ordered sums of squares associated with estimated effects of  $R_g$  for gamma-beta distortion, Weibull(2,2) distribution.

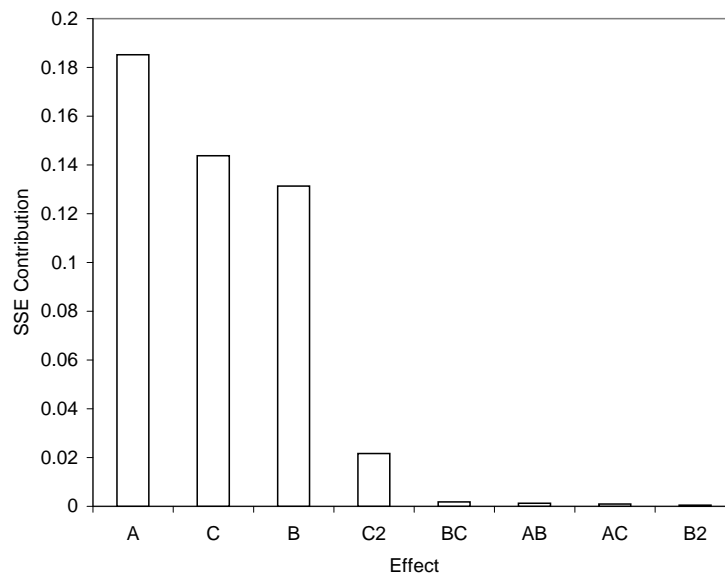


Figure 4.3 Ordered sums of squares associated with estimated effects of  $R_g$  for gamma-beta distortion, triangular(1,7,4) distribution.

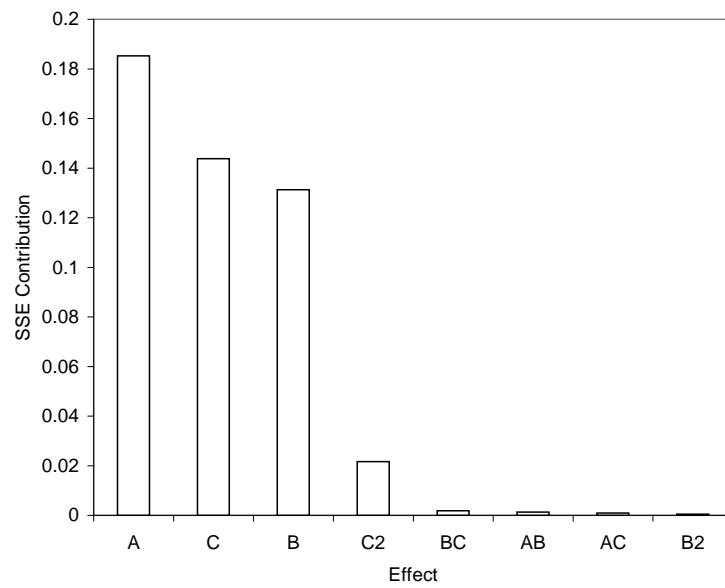


Figure 4.4 Ordered sums of squares associated with estimated effects of  $R_g$  for gamma-beta distortion, uniform(1,7) distribution.

Using eight effects in each model (excluding only  $A^2$ ), for every distribution the coefficient of multiple determination,  $R^2$ , was equal to 0.9991 and the  $R^2_{\text{adj}}$  (which takes into account the number of effects included in the model) was 0.99879. The fact that the  $R^2$  and  $R^2_{\text{adj}}$  values are so similar, and so close to unity, indicates that insignificant terms have been correctly excluded and that the model almost entirely accounts for the variability in the design region. Hence, these models would be very useful to the decision maker in navigating the design region. The specific quadratic models produced are included in Table 4.4.

Table 4.4 Summary of model equations for selected distributions, gamma-beta distortion, in terms of actual factors.

Distribution	Final Predictor Equation
Exponential	$\hat{R}_g = 1.740 - 1.031A + 0.593B - 0.210C - 0.200B^2 + 0.028C^2 + 0.344AB - 0.042AC + 0.044BC$
Weibull	$\hat{R}_g = 1.742 - 1.033A + 0.594B - 0.211C - 0.200B^2 + 0.028C^2 + 0.344AB - 0.042AC + 0.044BC$
Triangular	$\hat{R}_g = 1.740 - 1.031A + 0.593B - 0.211C - 0.200B^2 + 0.028C^2 + 0.344AB - 0.042AC + 0.044BC$
Uniform	$\hat{R}_g = 1.740 - 1.031A + 0.593B - 0.211C - 0.200B^2 + 0.028C^2 + 0.344AB - 0.042AC + 0.044BC$

Figure 4.5 shows a method of representing the equations in Table 4.4 graphically. To create and use a figure such as this, the decision maker must decide to “discretize” one of the variable parameters. In the figure the  $b$  parameter has been fixed at two distinct values, 1.1 and 1.5. The decision maker first chooses one of the two values of  $b$ ; if, in this case, 1.5 is chosen, then the solid line figure is used, and if 1.1 is chosen the dotted line figure is used. From that entry point, the user locates the continuous values for the  $a$  and  $c$  parameters within the applicable  $b$  “framework,” then moves horizontally left and uses the vertical axis to establish the value of  $R_g$ . As an example, if  $b = 1.5$  is selected (solid lines), then also choosing  $a = 0.6$  and  $c = 1.2$  yields  $R_g \approx 1.7$ .

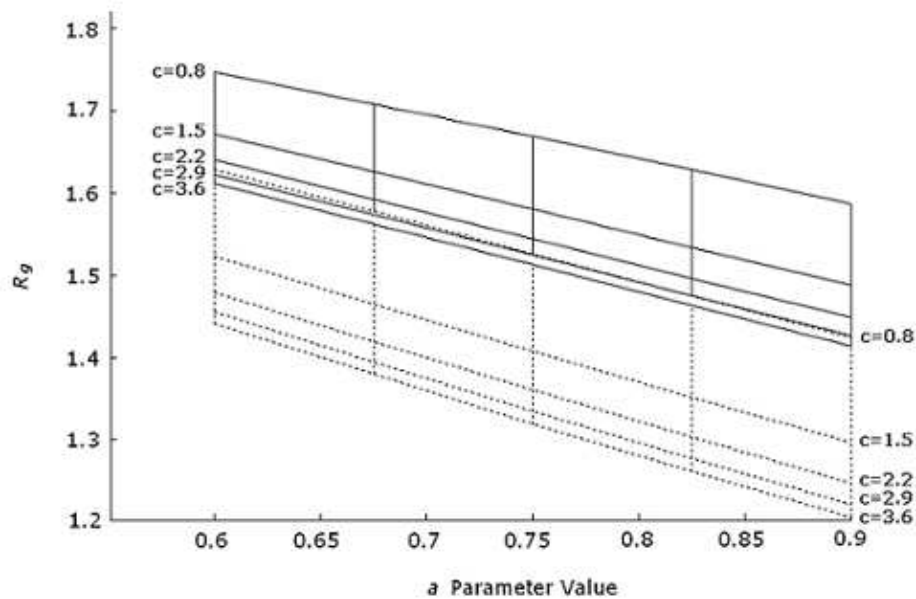


Figure 4.5 Graphical representation of gamma-beta effects on  $R_g$  for exp(3.5) distribution, holding parameter  $b$  constant (solid is  $b = 1.5$ ,  $\dots$  is  $b = 1.1$ ).

Given fixed distortion parameter values, the gamma-beta distortion produces the highest  $R_g$  values and maximizes the ability to “tweak” the density function by combining the effects of the single-parameter distortions (these effects will be discussed in detail shortly). However, the inability to calculate the associated expectations limits its use for risk analysis to only those (perhaps isolated) scenarios where the decision-maker is more interested in how the severity density function is shaped than the change in expectation. Thus, further study of the single-parameter distortions is more important to the field of risk analysis.

#### ***4.5 Effect of Single-Parameter Distortions***

Prior to beginning the parametric study of the single-parameter distortions, an analytical basis for the work was prescribed. For each combination of distribution and distortion (except the gamma-beta distortion), an attempt was made to establish an explicit expression for the (distorted) expectation risk measure. These results are recorded in Tables 4.5 through 4.8. Where results were intractable, the corresponding entry for the distorted expectation includes an integral. In all of the single-parameter distortion cases, numerical results for the expectation were achievable even where the analytical expectation was intractable. As a cross-check, numerical results were compared to the theoretical ones in all cases, and consistent accuracy to at least four decimal places was observed.

For the exponential distribution considered in Table 4.5, the survivor function is

$$S(x) = e^{-\lambda x}, \text{ where } x \geq 0, \lambda > 0, \quad (4.3)$$

and the undistorted expectation is  $\mu_0 = 1/\lambda$ .

Table 4.5 Summary of distortion and risk measure ( $X \sim \exp(\lambda)$ ).

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
$g_{PH}$	$e^{-\lambda ax}$	$(\lambda a)^{-1}$
$g_{DP}$	$1 - (1 - e^{-\lambda x})^b$	$\int_0^\infty [1 - (1 - e^{-\lambda x})^b] dx$
$g_{EX}$	$\frac{1 - \exp(-e^{-\lambda x}/c)}{1 - \exp(-1/c)}$	$\int_0^\infty \frac{1 - \exp(-e^{-\lambda x}/c)}{1 - \exp(-1/c)} dx$

For the Weibull distribution considered in Table 4.6, the survivor function is

$$S(x) = \exp((-x/\theta)^\beta), \text{ where } x \geq 0, \beta > 0, \theta > 0, \quad (4.4)$$

and the undistorted expectation is  $\mu_0 = \frac{\theta}{\beta} \Gamma(\frac{1}{\beta})$ , where  $\Gamma(\cdot)$  is the gamma function.

Table 4.6 Summary of distortion and risk measure ( $X \sim \text{Weib}(\beta, \theta)$ ).

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
$g_{PH}$	$e^{a(-x/\theta)^\beta}$	$\frac{\theta}{\beta \sqrt[\beta]{a}} \Gamma(\frac{1}{\beta})$
$g_{DP}$	$1 - (1 - e^{(-x/\theta)^\beta})^b$	$\int_0^\infty [1 - (1 - e^{(-x/\theta)^\beta})^b] dx$
$g_{EX}$	$\frac{1 - \exp(-e^{(-x/\theta)^\beta}/c)}{1 - \exp(-1/c)}$	$\int_0^\infty \frac{1 - \exp(-e^{(-x/\theta)^\beta}/c)}{1 - \exp(-1/c)} dx$

For the triangular distribution considered in Table 4.7, the survivor function is

$$S(x) = \begin{cases} 1, & x < \theta_1 \\ 1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}, & \theta_1 \leq x \leq m \\ \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}, & m < x \leq \theta_2 \\ 0, & x > \theta_2, \end{cases} \quad (4.5)$$

where  $\theta_1 \leq x \leq \theta_2$ ,  $\theta_1 \leq m \leq \theta_2$ , and  $\theta_1 < \theta_2$ ; the undistorted expectation is  $\mu_0 = (\theta_1 + \theta_2 + m)/3$ .



Table 4.7 Summary of distortion and risk measure ( $X \sim \text{tria}(\theta_1, \theta_2, m)$ ).

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
$g_{PH}$	$\left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a, \theta_1 \leq x \leq m$ $\left(\frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^a, m < x \leq \theta_2$	$\int_{\theta_1}^m \left(1 - \frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^a dx$ $+ \frac{(\theta_2-m)^{a+1}}{(2a+1)(\theta_2-\theta_1)^a}$
$g_{DP}$	$1 - \left(\frac{(x-\theta_1)^2}{(\theta_2-\theta_1)(m-\theta_1)}\right)^b, \theta_1 \leq x \leq m$ $1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b, m < x \leq \theta_2$	$m - \theta_1 - \frac{(m-\theta_1)^{b+1}}{(\theta_2-\theta_1)^b(2b+1)} +$ $\int_m^{\theta_2} \left[1 - \left(1 - \frac{(\theta_2-x)^2}{(\theta_2-\theta_1)(\theta_2-m)}\right)^b\right] dx$
$g_{EX}$	$\frac{1 - \exp\left(\frac{-1}{c} + \frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1 - \exp(-1/c)}, \theta_1 \leq x \leq m$ $\frac{1 - \exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1 - \exp(-1/c)}, m < x \leq \theta_2$	$\int_{\theta_1}^m \frac{1 - \exp\left(\frac{-1}{c} + \frac{(x-\theta_1)^2}{c(\theta_2-\theta_1)(m-\theta_1)}\right)}{1 - \exp(-1/c)} dx$ $+ \int_m^{\theta_2} \frac{1 - \exp\left(\frac{-(\theta_2-x)^2}{c(\theta_2-\theta_1)(\theta_2-m)}\right)}{1 - \exp(-1/c)} dx$

For the uniform distribution on  $[\theta_1, \theta_2]$  considered in Table 4.8, the survivor function is

$$S(x) = 1 - \frac{x - \theta_1}{\theta_2 - \theta_1}, \text{ where } \theta_1 \leq x \leq \theta_2, \theta_1 < \theta_2; \quad (4.6)$$

the undistorted expectation is  $\mu_0 = (\theta_1 + \theta_2)/2$ .

Table 4.8 Summary of distortion and risk measure ( $X \sim \text{unif}(\theta_1, \theta_2)$ ).

Distortion	$\hat{S}(x)$	$\hat{E}[X]$
$g_{PH}$	$\left(1 - \frac{x - \theta_1}{\theta_2 - \theta_1}\right)^a$	$(\theta_2 - \theta_1) \left(\frac{1}{a+1}\right)$
$g_{DP}$	$1 - \left(\frac{x - \theta_1}{\theta_2 - \theta_1}\right)^b$	$(\theta_2 - \theta_1) \left(\frac{b}{b+1}\right)$
$g_{EX}$	$\frac{1 - \exp(-(1 - \frac{x - \theta_1}{\theta_2 - \theta_1})/c)}{1 - \exp(-1/c)}$	$(\theta_2 - \theta_1) \left(\frac{1 - c + ce^{-1/c}}{1 - e^{-1/c}}\right)$

## 4.6 Graphical Results

All of the chosen distortion functions can be shown to have a different effect on the probability distributions (densities) and expectation risk measures under study. The impacts of each of the four distortion functions on each of the four selected probability distributions are examined individually in the following sections.

### 4.6.1 Distortion Effects: Exponential Distribution

Figure 4.6 shows the effect of applying the maximum distortion amounts from Table 4.3 (i.e.,  $a = 0.6$ ,  $b = 1.5$ , and  $c = 0.8$ ) to the  $\exp(3.5)$  density; hence, each

of the single-parameter distortions can individually shift approximately 30% of the density in the p.d.f. to the right, as previously described. The undistorted exponential density is depicted by the solid line. Among the single-parameter distortions, the proportional hazard (PH) distortion clearly has the greatest effect on the tail of the distribution, thickening it considerably. The dual power (DP) distortion, conversely, while still thickening the right tail slightly, has a much more noticeable effect on the left side of the distribution, pushing the mode away from zero. The exponential (EX) distortion can best be described as a combination of the effects of the PH and DP. Applying all of the distortion parameters at once via the gamma-beta (GB) distortion, about 75% of the density is moved to the right of the original median, and the effects on both the left and right sides of the density are drastic. The right tail is heavily thickened, the relative frequencies of the lesser severities are cut by nearly two-thirds, and the mode is pushed out from zero to approximately 0.3.

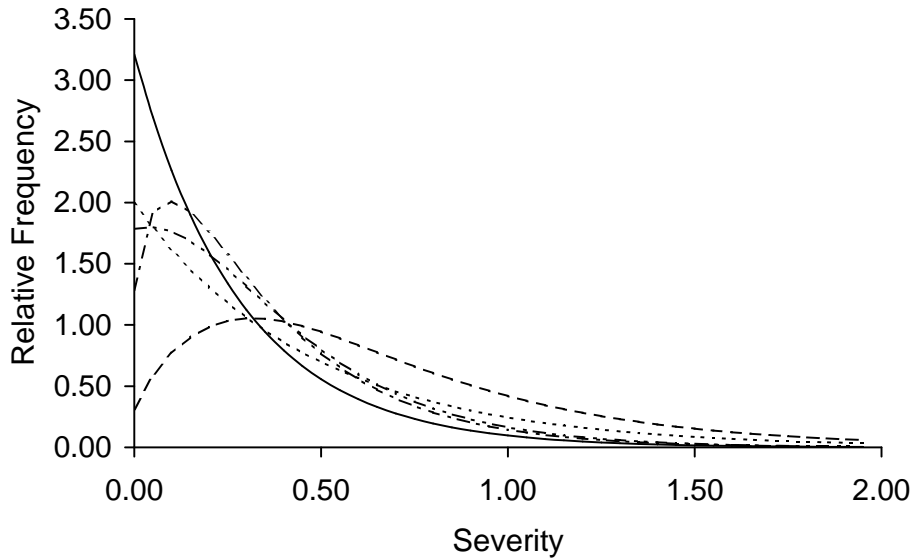


Figure 4.6 Relative frequency density for severity, exponential(3.5) distribution, given distortion parameters  $a = 0.6$ ,  $b = 1.5$ , and  $c = 0.8$  (solid is no distortion, — — — GB, . . . . PH, — · — DP, — · — EX) .

Figure 4.7 shows the effects of the single-parameter distortions on the expectation risk measure. “Coded” variables are used so that all three of the distortion

effects can be observed simultaneously; in this technique, the parameter value resulting in the greatest distortion is assigned the value -1, and the parameter value resulting in the least distortion is assigned +1. Table 4.9 below summarizes the coding scheme; note that the  $b$  parameter values must be “reversed” from what is logical in order to make the plot consistent, i.e., to show more effect on the left and less on the right.

Table 4.9 Linearly coded distortion parameter values for expectation plots.

Distortion Parameter	Actual Value	Coded Value
$a = [0.525, 0.975]$	0.525	-1
	0.75	0
	0.975	+1
$b = [1, 1.6]$	1.6	-1
	1.3	0
	1	+1
$c = [0.1, 4.3]$	0.1	-1
	2.2	0
	4.3	+1

From Figure 4.7, the most variable effect on the expectation risk measure comes from the EX distortion. While the effects of the PH and DP distortions are nearly linear over the selected intervals, the EX distortion shows a very large effect over

$(-0.5, -1)$ . This is primarily due to the *amount* of density the EX is shifting by the time  $c \rightarrow -1$ : while the PH and DP distortions ( $a$  and  $b$  parameters) are shifting 39% and 34% of the original density at their -1 values, respectively, by the time  $c = 0.1$  in actual value (-1 in coded value) the EX distortion is shifting over 98% of the original density to the right.

Thus the difference between using the limits specified in Table 4.3 and the wider ones from Table 4.9 is clearly significant. Once again, the values from Table 4.3 were selected to match up (to the greatest possible extent) the *amount* of density (in percent) each distortion function could shift; these tighter limits are depicted in Figure 4.7 as the severity range between  $[-\frac{2}{3}, \frac{2}{3}]$ . From Figure 4.6 (and to some extent 4.7) it is clear that, given essentially equal percentage shifts in density, the PH distortion has the greatest effect on the right tail of the exponential distribution, and generally the expectation. However, outside of that range, control over the amount of density being shifted is surrendered. As a rule of thumb, the user should note that if only a limited percentage of density shift is desired, the PH distortion *generally* gives the “biggest bang for the buck” *within the region under study*.

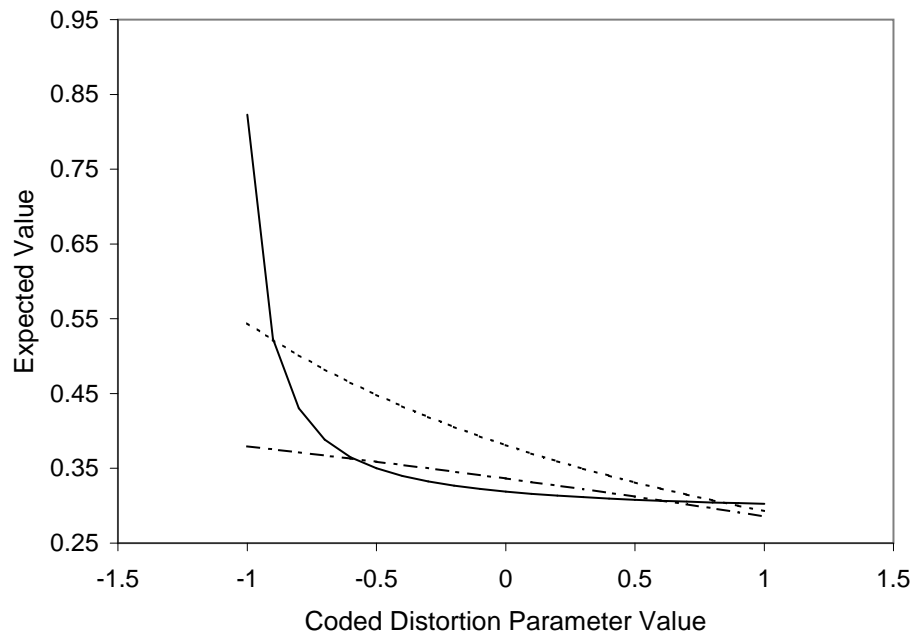


Figure 4.7 Expected value versus coded distortion parameters, exponential(3.5) distribution, given distortion parameter ranges  $a = [0.525, 0.975]$ ,  $b = [1, 1.6]$ , and  $c = [0.1, 4.3]$  ( $\cdots$  PH,  $- \cdot -$  DP, solid is EX).

#### 4.6.2 Distortion Effects: Weibull Distribution

Figure 4.8 shows the effects of the various distortion functions on the Weibull(2,2) distribution, again using the maximum distortion amounts from Table 4.3. The undistorted Weibull density is depicted by the solid line. As in the case of the exponential distribution, the PH distortion thickens the right tail to the greatest extent among the single-parameter distortions; the DP has the least effect on the right tail, and the EX is between the others. As before, the GB shifts about 75% of the density to the right using these parameter selections and vastly changes the appearance of the distribution tail.

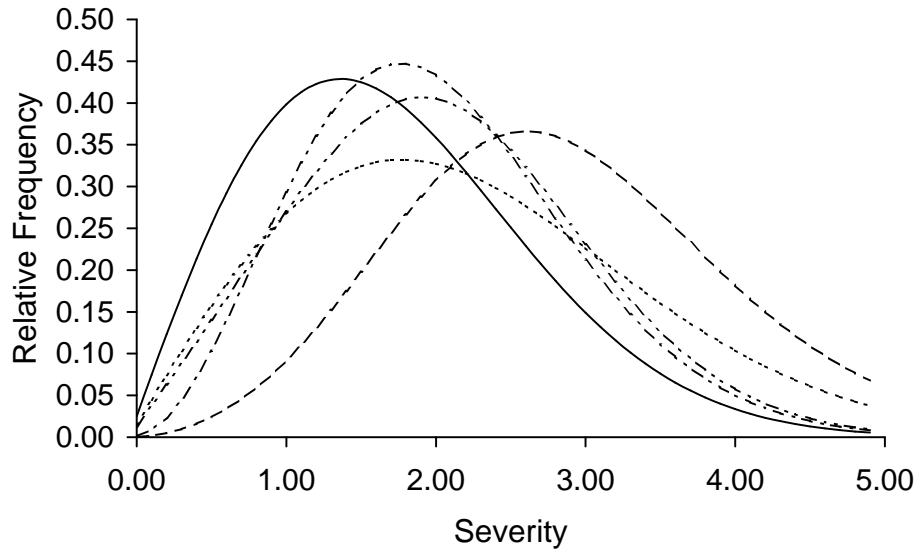


Figure 4.8 Relative frequency density for severity, Weibull(2,2) distribution, given distortion parameters  $a = 0.6$ ,  $b = 1.5$ , and  $c = 0.8$  (solid is no distortion, — — — GB, ··· PH, - · - · DP, - · - - EX) .

The comments associated with Figure 4.7 transfer in full to Figure 4.9.

#### 4.6.3 Distortion Effects: Triangular Distribution

The triangular(1,7,4) distribution is similarly shown as the solid line in Figure 4.10. Applying the single-parameter distortions to the triangular density, there is

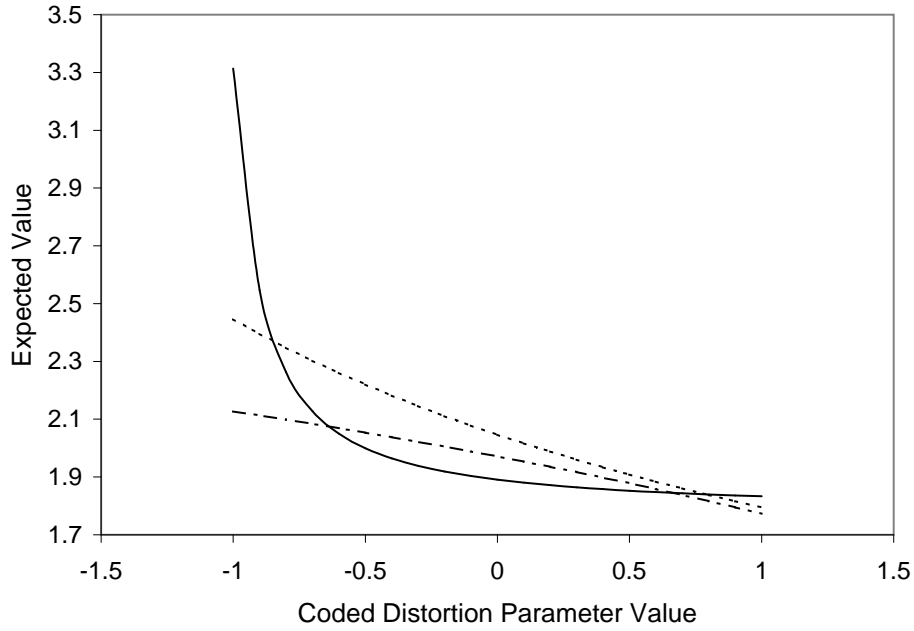


Figure 4.9 Expected value versus coded distortion parameters, Weibull(2,2) distribution, given distortion parameter ranges  $a = [0.525, 0.975]$ ,  $b = [1, 1.6]$ , and  $c = [0.1, 4.3]$  ( $\cdots$  PH,  $-\cdot-\cdot$  DP, solid is EX).

consistency in the observed distortion effects from the exponential and Weibull cases. The effect on the right side of the distribution is again greatest with the PH distortion, followed in order by the EX and DP. After application of the GB distortion, the density is radically shifted and scarcely resembles a triangular density.

The comments associated with Figure 4.7 transfer in full to Figure 4.11.



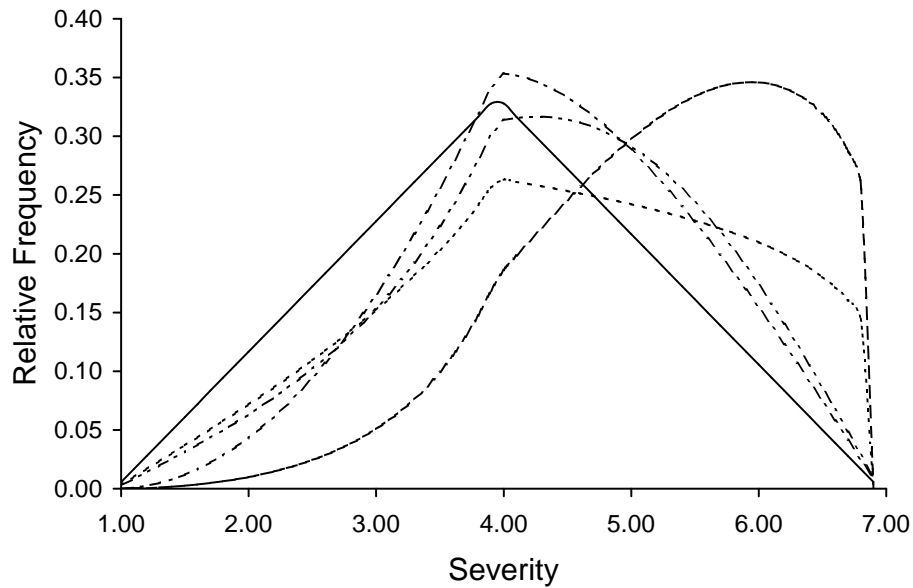


Figure 4.10 Relative frequency density for severity, triangular(1,7,4) distribution, given distortion parameters  $a = 0.6$ ,  $b = 1.5$ , and  $c = 0.8$  (solid is no distortion, - - - GB,  $\cdots$  PH, -  $\cdot$  - DP, -  $\cdot$  - EX) .

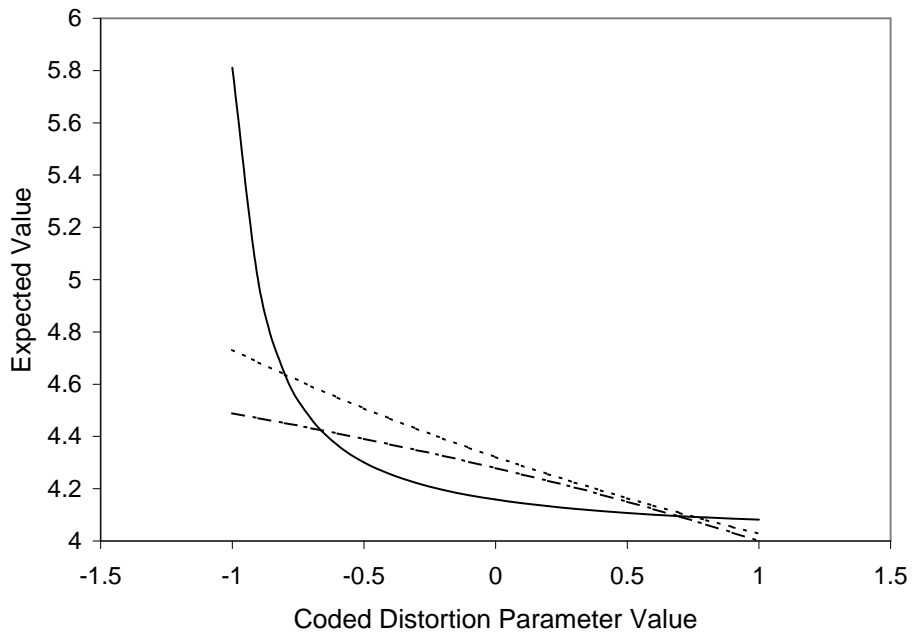


Figure 4.11 Expected value versus coded distortion parameters, triangular(1,7,4) distribution, given distortion parameter ranges  $a = [0.525, 0.975]$ ,  $b = [1, 1.6]$ , and  $c = [0.1, 4.3]$  ( $\cdots$  PH, -  $\cdot$  - DP, solid is EX).

#### 4.6.4 Distortion Effects: Uniform Distribution

Figure 4.12, as with the previous plots, uses the distortion parameters  $a = 0.6$ ,  $b = 1.5$ , and  $c = 0.8$ . The classic  $\text{unif}(1,7)$  is observed as the solid line. In this case, the GB distortion has the effect of nearly eliminating the relative frequency of the lightest severities altogether while simultaneously increasing those of the harshest severities approximately fourfold. Among the single-parameter distortions, the PH emphasizes the far right tail severities most significantly, and the DP and EX distortions retain their roles from the previous test cases. The DP, EX, and PH, in that order, have the largest effect on reducing the left side of the density; this is again consistent with previous results.

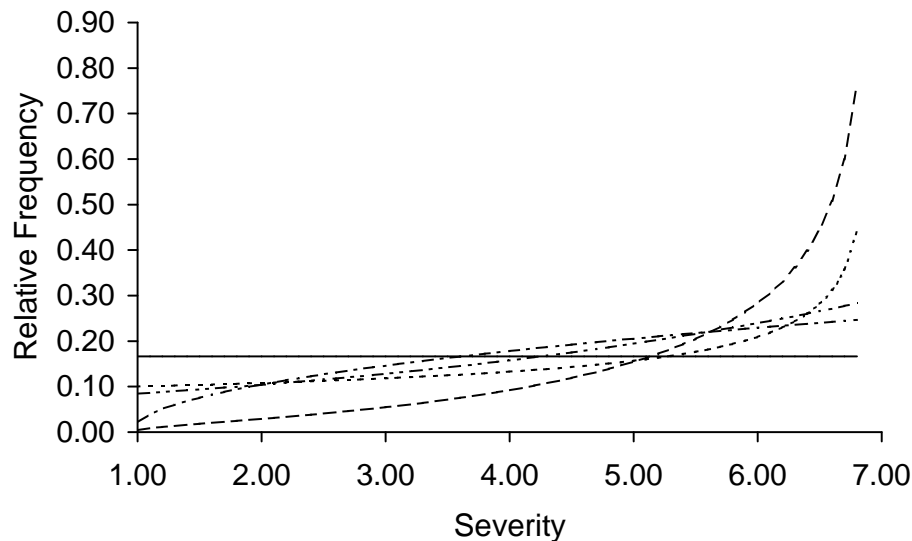


Figure 4.12 Relative frequency density for severity,  $\text{uniform}(1,7)$  distribution, given distortion parameters  $a = 0.6$ ,  $b = 1.5$ , and  $c = 0.8$  (solid is no distortion, — — — GB, . . . . PH, — · —· DP, — · · — EX) .

The comments associated with Figure 4.7 transfer in full to Figure 4.13.

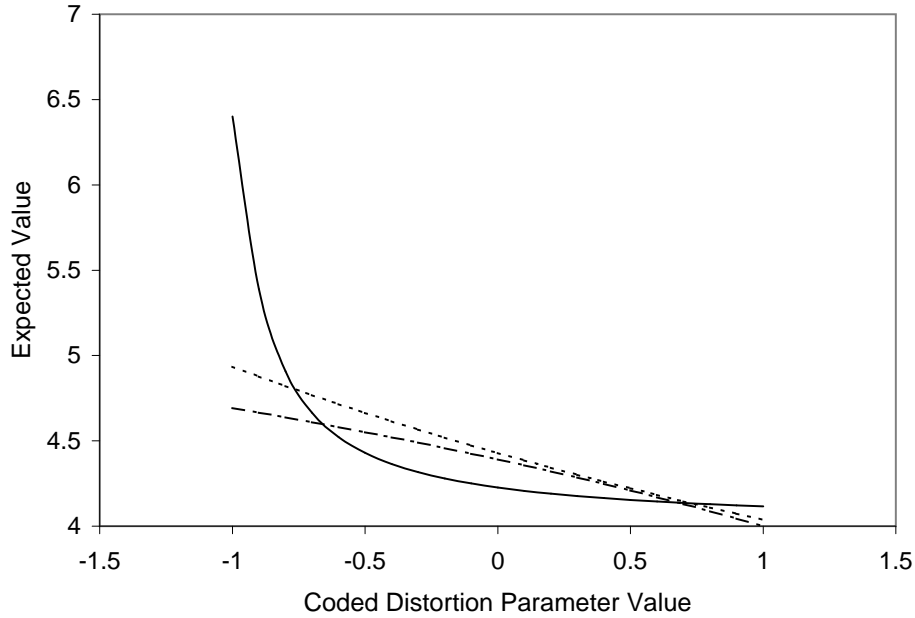


Figure 4.13 Expected value versus coded distortion parameters, uniform(1,7) distribution, given distortion parameter ranges  $a = [0.525, 0.975]$ ,  $b = [1, 1.6]$ , and  $c = [0.1, 4.3]$  ( $\cdots$  PH,  $- \cdot - \cdot$  DP, solid is EX).

#### 4.7 Distortion Effectiveness and Efficiency

The primary risk measure under study is the expectation of the severity random variable. Recall that expectation has a drawback in that it serves only as the center of mass for the distribution, meaning it is generally “dampened out” by the values with the greatest relative frequency.

However, if expectation is the risk measure of choice, distortion functions can provide the decision maker with the ability to control expectation to minute, yet predictable, degrees. In choosing a distortion function to apply to an SME-specified severity distribution, the decision maker would like to know how *effective* each candidate distortion function/parameter combination is in increasing the expectation. After applying each combination and determining the distorted expectations ( $\mu_g$ ), these values can be directly compared to determine which distortion has the greatest effect on that particular distribution’s mean.

To develop the idea further, we can normalize all of the  $\mu_g$ 's to the original undistorted expectation,  $\mu_0$ . The resulting ratio,

$$K = \mu_g / \mu_0, \quad (4.7)$$

can, for example, be used to directly compare a unique distortion function/parameter pairing *over different distributions*, measuring that pairing's *effectiveness* in changing each distribution's expectation as a percentage increase. (Since the distortion functions discussed here only move density to the right, then  $K \geq 1$ , where  $K = 1$  means no distortion has been applied.) Similarly, two different distortion function/parameter combinations applied to two dissimilar severity distributions but with equal resulting  $K$  values are judged equally *effective* in distorting (increasing) the expectations.

However, we have observed clear contrasts in the way different distortion function/parameter pairs shift density. As applied to a single severity distribution, one combination may require significant density shift before its  $K$  matches that of another pairing which has a greater effect on the distribution's tail. The classic examples here are the PH and DP distortions: the PH accumulates density in the right tail while the DP accumulates it closer to the mode, so the PH generally has a greater effect on expectation. Of course, a measure to reflect the amount of density shift has already been established, namely  $R_g$  (although the measure can only be employed over a segment of each distortion's parameter range(s), as previously described).

Considerable benefit is obtained by combining  $K$  and  $R_g$  into a single measure. Using a ratio of the two,

$$\frac{K}{R_g} = \frac{\% \text{ change in } \mu}{\% \text{ change in density}} = \frac{\Delta\mu}{\Delta\text{density}}, \quad (4.8)$$

which can be considered the *efficiency* of a distortion function/parameter pairing (not to be confused with statistical efficiency, as related to parameter estimation).

Intuitively, if a distortion function/parameter combination has a large *effect* on the distribution mean while shifting only a small amount of density, then that pairing is highly *efficient* when applied to the given distribution.

A logical question to ask next might be, “Why would a decision maker care about the amount of density being shifted? Why isn’t the *effectiveness* of the distortion function/parameter combination all he or she needs to know in making a selection?” Note that without the *efficiency* measure, there would be no need to distinguish between two pairings with identical *effectiveness*; the decision maker would feel that one is just as good as the other, even though the underlying distribution is being changed in an entirely different manner depending on the choice. As an example, consider Figure 4.14, which shows an undistorted Weibull(2,2) distribution along with its PH ( $a = 0.2$ ) and DP ( $b = 31$ ) distortions. Both of the distorted distributions have  $K \approx 2.24$ , but the densities are hardly similar.

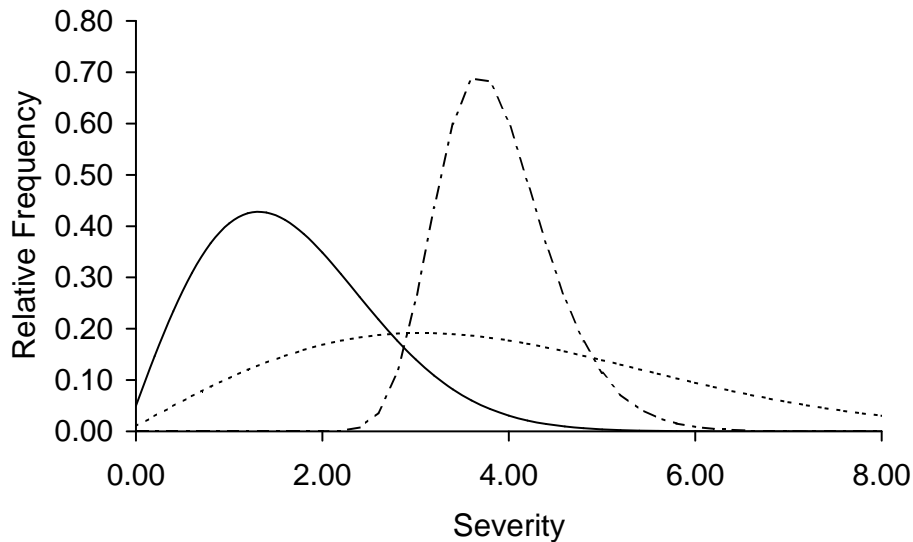


Figure 4.14 Relative frequency for severity, Weibull(2,2) distribution, given distortion parameters  $a = 0.2$  and  $b = 31$  (solid is no distortion,  $\cdots$  PH,  $- \cdot -$  DP).

Furthermore, assuming that the decision maker wants the “correct” model but is forced to distort the distribution due to the intolerable far right tail risks involved,

the decision maker should care a great deal about how much density is being shifted to achieve a desired increase in expectation. The organization, be it a business or a government agency, has asked the SMEs for their opinions because the SMEs have expertise the decision maker lacks; the decision maker may have hand-picked the individuals who are providing the base severity distribution that he/she wishes to distort. For every increase in  $R_g$ , the decision maker is taking an additional “step” away from the recommendations of the SMEs. As a clear example of the power of distortion, again consider Figure 4.14, where the distortions have changed the SME-provided Weibull distribution into two radically different ones. Thus it seems likely that the decision maker would prefer one of two possible courses of action in choosing a distortion function/parameter combination:

1. achieve the maximum amount of increase in the expectation while affecting the base severity distribution by (no more than) a specified amount (say 20%); or
2. achieve a fixed change in expectation, but affect the base severity distribution as little as possible.

In either case, efficiency is the measure which provides the “best” answer.

Table 4.10 records the *efficiency* and *effectiveness* measures for the severity distributions and single-parameter distortions studied in this chapter. For each distribution, the first line shows the distorted expectation,  $\mu_g$ , for each distortion function/parameter setting; the second line is the percentage of density shifted  $R_g$ ; the third line is the normalized distorted mean,  $K$ ; and the last line is the *efficiency*,  $K/R_g$ . In general, as the amount of distortion is increased the efficiency is *reduced*. There are three exceptions to the general rule, however, and the efficiency is highlighted in **bold** for those three. Specifically, in the case of the exponential distribution, an increase in distortion results in an *increase* in efficiency when using the PH and EX distortions. For the PH distortion applied to the Weibull distribution, efficiency at first decreases as distortion is increased, then changes course and

begins to increase again; a brief investigation to verify this result showed that the least efficiency occurred at about  $a = 0.72$ .

Table 4.10 Summary of effectiveness and efficiency measures for distribution and single-parameter distortion combinations.

Distortion $\rightarrow$	PH			DP			EX		
Measure $\downarrow$	$a = 0.9$	$a = 0.75$	$a = 0.6$	$b = 1.1$	$b = 1.3$	$b = 1.5$	$c = 3.6$	$c = 2.2$	$c = 0.8$
Exponential(3.5), $\mu_0 = 0.285714$									
$\mu_g$	0.3175	0.3810	0.4762	0.3036	0.3363	0.3658	0.3058	0.3189	0.3791
$R_g$	1.0718	1.1892	1.3193	1.0670	1.1877	1.2929	1.0693	1.1131	1.3027
$K$	1.1111	1.3333	1.6667	1.0625	1.1772	1.2803	1.0704	1.1161	1.3270
$K/R_g$	<b>1.0367</b>	<b>1.1212</b>	<b>1.2633</b>	0.9958	0.9911	0.9903	<b>1.0010</b>	<b>1.0027</b>	<b>1.0186</b>
Weibull(2,2), $\mu_0 = 1.772454$									
$\mu_g$	1.8683	2.0467	2.2882	1.8448	1.9713	2.0788	1.8449	1.8911	2.0971
$R_g$	1.0719	1.1895	1.3199	1.0670	1.1879	1.2932	1.0694	1.1133	1.3032
$K$	1.0541	1.1547	1.2910	1.0408	1.1122	1.1729	1.0408	1.0669	1.1831
$K/R_g$	<b>0.9834</b>	<b>0.9707</b>	<b>0.9781</b>	0.9755	0.9362	0.9069	0.9733	0.9584	0.9079
Triangular(1,7,4), $\mu_0 = 4.000$									
$\mu_g$	4.1163	4.3218	4.5777	4.1033	4.2793	4.4246	4.0971	4.1586	4.4275
$R_g$	1.0718	1.1892	1.3193	1.0670	1.1877	1.2929	1.0693	1.1131	1.3027
$K$	1.0291	1.0804	1.1444	1.0258	1.0698	1.1062	1.0243	1.0396	1.1069
$K/R_g$	0.9602	0.9086	0.8675	0.9614	0.9007	0.8556	0.9579	0.9340	0.8497
Uniform(1,7), $\mu_0 = 4.000$									
$\mu_g$	4.1579	4.4285	4.7500	4.1428	4.3913	4.6000	4.1387	4.2265	4.6093
$R_g$	1.0718	1.1892	1.3193	1.0670	1.1877	1.2929	1.0693	1.1131	1.3027
$K$	1.0395	1.1071	1.1875	1.0357	1.0978	1.1500	1.0347	1.0566	1.1523
$K/R_g$	0.9699	0.9310	0.9001	0.9707	0.9243	0.8895	0.9676	0.9492	0.8846

Using Table 4.10, some general rules (within the bounds of this study) can be established for selecting a distortion function to apply to a distribution. Recall that a decision maker would likely be interested in either (1) achieving the largest possible increase in the mean given a specified maximum shift in density, or (2) shifting the density by the smallest amount required to achieve a specified increase in mean. Using Table 4.10, some answers may be available when objective (1) is of primary importance. Table 4.11 was created from Table 4.10 by comparing *efficiency* across categorized values of  $R_g$ . For example, looking at the triangular distribution in Table 4.10, the low distortion efficiency values are 0.9602 for the PH ( $a = 0.9$ ), 0.9614 for the DP ( $b = 1.1$ ), and 0.9579 for the EX ( $c = 3.6$ ). Since the DP value is the highest, this was entered into the appropriate cell of Table 4.11. Thus in the

case of objective (1) when assuming a triangular distribution, the DP distortion is the most efficient (although the values are relatively close in this case).

In examining Table 4.11, note once again that the difference in the  $R_g$  values between the PH and DP distortions ( $R_g \approx 1.19$ ) and the EX distortion ( $R_g \approx 1.11$ ) at the “medium” distortion level could be significant in the final selection of a distortion function at that level. Also note that decision maker objective (2) could be answered just as easily as objective (1), but the original response surface study which facilitated the distortion parameter choices would have had to fix the distorted *expectations* rather than the *amount of density shift* being applied.

Table 4.11 Preferred distortion functions, by *efficiency*, for selected distributions.

Distribution	Low Distortion (0-10%)	Moderate Distortion (11-20%)	Heavy Distortion (21-30%)
Exponential(3.5)	PH	PH	PH
Weibull(2,2)	PH	PH	PH
Triangular(1,7,4)	DP	EX	PH
Uniform(1,7)	DP	EX	PH



#### 4.8 Guidelines for Selection of Distortion/Distribution Combinations

To complete the goals set out at the beginning of the chapter, we now provide some general guidelines for selecting distribution/distortion function combinations. All of the following observations are valid within the limits of the study.

1. Among the selected distortions, given fixed values for distortion parameters  $a$ ,  $b$ , and  $c$ , the gamma-beta distortion has the greatest ability to shift density, regardless of the severity distribution to which it is applied. However, the overall effect is less than additive over the single-parameter distortions, meaning  $R_{GB} < R_{PH} + R_{DP} + R_{EX}$ . Furthermore, the inability to compute the distorted expectation using simple numerical methods makes the GB distortion a less than appealing choice for risk analysis.
2. When the SME suggests an exponential or Weibull distribution for the risk scenario, the PH distortion is drastically more *efficient* than the DP or EX. In the case of the exponential distribution, the PH also leaves the mode in place at zero, while other distortions “pull” the mode away from zero.
3. For the triangular and uniform distributions, no distortion appears to be as totally dominant (in efficiency) as the PH is for the exponential and Weibull. For each of these “bounded” distributions, the DP distortion is the most efficient in cases where only a small amount of distortion is required; *questionably*, the EX is more efficient in the vicinity of  $R_g = 1.15$ ; and the PH is most efficient when larger amounts of distortion are required.
4. If more than just the expectation is to be pulled from the distorted distribution (e.g., the variance may well be of concern), then the DP and EX distortions may be preferred over the PH. Particularly in the case of the Weibull and triangular distributions, the DP builds up the area around the mean, likely reducing the impact on variance.

5. As stated in Chapter 2, the  $b$  parameter of the DP distortion has the advantage of being interpretable, specifically as the expected value of the worst outcome when  $b$  samples are taken from the random variable [31]. If the decision maker appreciates this interpretability but wishes to use either the PH or EX distortion, a value of  $b$  can be found which results in a DP match in  $\mu_g$  to the specified  $a$  or  $c$  parameter. In this manner, the interpretability can be “loaned” to the PH and EX distortions through a single extra step.

## 4.9 Summary

This chapter began by selecting a subset of four from the gamma-beta family of six distortion functions. The selections were motivated primarily by frequency of references in the literature and the ability to compute expectations, either analytically or numerically (except in the case of the gamma-beta distortion itself).

The examination of the gamma-beta distortion and the effects that examination had on the remainder of the parametric analysis were discussed next. In particular, the face-centered cube approach used in the  $3^k$  factorial design prompted the creation of the  $R_g$  measurement, which then motivated the discussion of *effectiveness* versus *efficiency* near the end of the chapter.

For the single-parameter distortions, in all cases it was observed that the PH distortion has the greatest effect on the right tail of the distribution, while the DP operated closest to the mode and the EX somewhere between the two. The importance of tracking the amount of density shifted in comparing distortions was highlighted in discussions using coded distortion parameter plots.

Finally, recommendations were made regarding distortion function selection based on the chosen severity distribution and the degree of density shift required (or requested) by the decision maker. An argument was made that the decision maker should seek to limit the amount of distortion applied to the distributions.

## 5. Numerical Results

Chapters 3 and 4 have described a methodology for selecting probability distributions and illustrated the effect of different distortion functions on those distributions. In this chapter, those ideas will be illustrated and applied to solve a resource allocation problem. First, the original “real-world” results from the Capabilities Review and Risk Assessment (CRRA) process will be reviewed, and some interpretation of them provided. Notional SME data, based on those interpretations, will then be used to select specific risk distributions. To complete the example, the *a priori* stated policies of a fictitious decision maker will be implemented using selected distortion functions.

### 5.1 *Declared Shortfalls in Air Force Capabilities*

In a December 17, 2003 press release, the USAF listed six critical shortfalls (identified by the CRRA) that the service needs to address in both current and future budget planning [28]. The six main points from this public release is summarized in Table 5.1.

We will assume that the far right column of the table identifies those numbered top-level capabilities of the Master Capabilities Library (MCL) which the USAF is targeting with the corresponding statement; note the prominent roles of capabilities 1, 2, and 3 (corresponding to surveillance and reconnaissance, intelligence, and command and control), and to some extent 4 (communications). Note also, for this notional interpretation, that capabilities 5, 6, and 7 receive only limited consideration, while capabilities 8 and 9 are not included at all among the current CRRA shortfalls.

Table 5.1 Recent capability shortfalls identified by USAF leadership.

CRRA Shortfall	Top-Level MCL Item(s)
<i>Base defense</i> : The service needs to clarify the roles and responsibilities between the Air Force and the other services.	7
<i>Battle space management</i> : The service must create a common operational picture and implement an effects-based approach to war planning.	1, 2, 3, 4
<i>Cargo airlift</i> : The Air Force should begin a formal review of requirements and prepare for possible force structure changes.	6
<i>Battle damage assessment</i> : The Air Force should build a toolkit and definitions for commanders to determine effects-based decisions.	1, 2, 3
<i>Fleeting and mobile targets</i> : The service must reduce the time it takes to find, track and destroy enemy forces.	1, 2, 3, 4, 5
<i>Global information grid</i> : The Air Force must create a massive system to collect, process, store, disseminate and manage information for war fighters, policy-makers and support personnel.	1, 2, 3, 4

## 5.2 Scenarios Reflecting the Range of Severities

Using the information from the third column of Table 5.1, assume that the number of “references” to each of the top-level capabilities roughly reflects the potential severity of having a shortfall in that area. For example, since MCL items 1, 2, and 3 receive the most mention in the table, assume that capability shortfalls in those areas will result in higher severities, given that a shortfall will result in a consequence.

In this section we will examine some real-world and notional manifestations of these shortfalls. To facilitate some of the upcoming discussion, the complete cross-reference of severity categories and descriptions, first mentioned in Chapter 1, is included in this chapter as Tables 5.2 and 5.3.

Table 5.2 CRRA severity categories and descriptions.

Consequence Category ↓	Severity Factor		
	Minor (1)	Modest (2)	Substantial (3)
Achievement of Objectives	All major objectives achieved. Strong initial strategy requires few/no adjustments. Objectives achieved on time.	All major objectives achieved. Strong initial strategy requires modest adjustments. Few operational delays. Few delays in achieving campaign objectives.	All major objectives achieved, but strategy adjustments required along the way. Some operations slowed. Achievement of a major objective delayed.
Friendly Casualties	Few citizens/troops killed or injured. Citizens overseas threatened.	Tens of citizens/troops killed or injured. Citizens overseas attacked and/or injured.	Hundreds of citizens/troops killed/injured. Citizens overseas attacked/taken hostage.
Friendly Capability	Air/land/sea/space control unchallenged. No combat losses. All mutual support requests fulfilled.	Superiority achieved in/over all areas on time; no holdout areas. Enemy capabilities do not disrupt any missions. Almost all requests for mutual support fulfilled.	Superiority in/over enemy territory delayed; a few holdout areas avoided. Enemy capabilities disrupt some missions. Most requests for mutual support fulfilled.
Friendly Infrastructure	No loss of critical infrastructure.	Local/limited damage to critical infrastructure. No regional damage or loss.	Local damage to critical infrastructure. No regional damage or loss.
Collateral Damage	Few to dozens killed or injured in collateral damage. Local damage/destruction to buildings/infrastructure.	Dozens to hundreds killed or injured in collateral damage. City-wide damage/destruction to buildings/infrastructure.	Hundreds to thousands killed or injured in collateral damage. Regional damage/destruction to buildings/infrastructure.
Enemy Escalation/ WMD	Enemy offensives stopped as they are started. No threats to friendly bases. Continuous monitoring of known CBRNE sources.	Enemy offensives stopped in their early stages. Direct, credible threats to friendly bases. Threat of CBRNE use/attack possible.	Enemy offensives make some gains before being driven back. A friendly base attacked and damaged. Credible threat of CBRNE use/attack.
U.S. National Integrity	No enemy advances toward US territory/airspace. No terror attacks/incidents on US territory.	No enemy advances toward US territory/airspace. No terror incidents on US territory.	Embassies fired on. Conventional enemy forces observe US territory/airspace; are prevented from encroaching. Terror attack with conventional arms/explosives on US territory.
U.S. Government Function	State or federal first responders may go on heightened alert. No recovery action(s) required.	State government(s) executes well prepared recovery actions. Federal government assistance not needed.	State government falters occasionally in executing recovery plans. Federal government assistance necessary.

Table 5.3 CRRA severity categories and descriptions (cont'd).

Consequence Category ↓	Severity Factor		
	Major (4)	Extensive (5)	Catastrophic (6)
Achievement of Objectives	One or more major objectives in jeopardy of not being achieved. Several major strategy adjustments required. Advances toward objectives slowed/stalled. Delayed achievement of campaigns major objectives.	One or more major objectives not achieved. Inadequate strategy requires many major adjustments. Advances toward objectives stalled. Major time pressures to achieve objectives and end campaign.	Major objectives not achieved. No strategy adjustments will allow objectives to be achieved. Time pressures force a decision to end the campaign without achieving objectives.
Friendly Casualties	Hundreds to thousands of citizens/troops killed/injured. Citizens overseas killed/taken hostage.	Thousands to tens of thousands of citizens/troops killed/injured. Citizens overseas killed/taken hostage.	Hundreds of thousands of citizens/troops killed/injured. Many citizens overseas killed/taken hostage.
Friendly Capability	Superiority in/over enemy territory not completely achieved; a few areas continuously avoided. A few unanswered challenges from enemy capabilities. Mutual support only for high priority needs.	Superiority in/over enemy territory limited in area or duration; some areas avoided. Some unanswered challenges from enemy capabilities. Mutual support very limited.	Superiority limited to friendly territory or achieved only for specific missions; significant areas avoided. Major unanswered challenges from enemy capabilities. Unable to provide mutual support.
Friendly Infrastructure	Local damage/loss of critical infrastructure. Regional infrastructure affected.	Some damage to friendly centers of gravity. Regional damage/loss of critical infrastructure.	Friendly centers of gravity damaged or destroyed. Widespread damage/loss of critical infrastructure.
Collateral Damage	Thousands to tens of thousands killed or injured in collateral damage. Regional damage/destruction to buildings/ infrastructure.	Tens of thousands killed or injured in collateral damage. Multi-region damage/destruction to buildings/ infrastructure.	Hundreds of thousands killed or injured in collateral damage. Widespread damage/destruction to buildings/infrastructure.
Enemy Escalation/ WMD	Enemy offensives make significant gains before being driven back. More than one friendly base attacked. Some CBRNE attacks, but we have adequate detection and warning.	Enemy offensives make significant gains. Widespread attacks on friendly bases. Some enemy use of CBRNE weapons. No warning for half the attacks; adequate warning for half the attacks.	Enemy offensives make gains we cannot counter. Widespread attacks on friendly bases. Widespread use of CBRNE weapons with no warning for most attacks. Detection occurs after attack.
U.S. National Integrity	Conflict is non-nuclear but involves terrorism, chemical, bio, or radiological strikes on US territory. Embassies occupied. Conventional enemy forces encroach upon US territory/airspace, but do not fire on it.	US survives as a nation. Active conflict where enemies fire on US territory/penetrate US airspace. Single nuclear strike on US territory. CBRNE incidents.	National survival threatened, loss of territorial integrity. Long term exhausting war. Entire nation focused on resolving conflict. Some nuclear strikes on US territory.
U.S. Government Function	Attack recovery is difficult. Federal government focuses on it above all else. State government focuses on it above all else. Federal government assistance required for response.	Federal and state governments struggle to cope with attack(s). Losing war would mean ideological and cultural realignment.	Survival of a functioning government is threatened. Losing war would mean ceding sovereignty or occupation.

### 5.2.1 “Minor” to “Modest” Severity Scenario

We will assume that senior USAF leaders categorize capability shortfalls in preparing, sustaining, and creating the force (MCL items 8 and 9) as holding the potential for primarily *minor* to *modest* severity. There are several possible reasons for the lack of emphasis on these areas. While shortfalls in MCL areas 8 and 9 certainly *could* affect achievement of objectives and capabilities of friendly forces, selective service (i.e., drafting of civilians into the military) is available as a fall-back position for force creation; the government would likely not allow a lack of volunteers to prevent the USAF from accomplishing its mission. Due to the overall wealth of the United States, the cost of training and equipping forces and supplying bases for operations has been a minimal hindrance to military objectives; the policy is simply not to send untrained or under-equipped forces into battle. The thrust into Iraq during Operation Iraqi Freedom (OIF) in 2003 faced some reported shortfalls in supplies and equipment, yet the available supplies and the technological superiority of friendly forces were sufficient to end the force-on-force campaign successfully and with little delay. These situations likely meet the criteria of *minor* or *modest* severity.

### 5.2.2 “Substantial” Severity Scenario

Shortfalls in force application, force projection, and force protection capabilities (MCL items 5, 6, and 7) will be assumed to result in *substantial* severity outcomes. A notional scenario (based on a real-world deployment) which fits this category occurred in conjunction with Operation Enduring Freedom (OEF) in Afghanistan. During that period, the USAF deployed 24 bombers to Andersen AFB, Guam, as a hedge against other potential enemies in the region deciding to take aggressive action while our attention was focused elsewhere. The primary concern for the USAF in terms of force protection, as identified by the December 2003 CRRA, is the clarification of base defense roles between the USAF and other services. If an enemy at that time had launched an attack from the sea with the intent of destroy-

ing the bombers deployed on Guam (perhaps using just a few incendiary projectiles fired from the sea with the intent of causing secondary explosions on the flightline), the U.S. Navy may or may not have been able to detect or stop the enemy before significant damage to the aircraft had taken place. In such an event, operations may have been disrupted, tens or even hundreds of servicemen and civilians killed on base, and achievement of objectives in the region delayed. All of these consequences fall into the range of *substantial* severity.

### 5.2.3 “Major” to “Extensive” Severity Scenario

In the December 2003 CRRA evaluation Table (5.1), communications is invoked three times among the six areas requiring improvement. Communications in the USAF sense deals (to a large extent) with hardware – the physical components of command and control (including computers). Due to the number of times communications is singled out on the list, we might assume that the USAF sees communication as potentially having more grave consequences than force application, force projection, or force protection, but not quite as harsh as intelligence, surveillance, and reconnaissance (ISR), and command and control (C<sup>2</sup>). As communications also includes the protection and assurance of data resident on computers and other communications systems, a straightforward example of a crisis with *major* to *extensive* severity is a large-scale “hacker” attack on a federal computer system where some critical infrastructure element is shut down. Recall the widespread power outages suffered in the northeastern U.S. during the summer of 2003, which resulted in temporary loss of critical regional infrastructure and were rampantly assumed to be the result of such an attack, even though this idea was later proven false.

### 5.2.4 “Extensive” to “Catastrophic” Severity Scenario

Again citing the December 2003 list, we will assume that the USAF believes that the greatest threats to national security will be caused by shortfalls in ISR and



command and control of forces (MCL items 1, 2, and 3). An example of the effects of shortfalls in ISR, albeit from more of a law enforcement perspective, is the 9/11 terrorist attacks using hijacked airliners. Since 9/11, the clear emphasis on change has involved the military and federal ISR systems and organizations, and the USAF plays a major role in providing those assets. The USAF and the DoD have built a vision for the battlefield of the future which centers around the ability to gather intelligence and transmit it quickly to military decision makers and forces in the field (this is known as network-centric warfare, or NCW – see [26] or other sources for further information). At the same time, advances in ISR systems can also aid intelligence professionals here at home in ascertaining the plans of adversaries before they are able to attack the homeland.

#### *5.2.5 Uniform Severity Scenario*

In some scenarios, all categories of severity could have an equal likelihood of occurrence. As an example, consider the illegal entry into the United States by a few persons from a terrorist organization with the intent of causing harm. The scope of what they might be able to accomplish could include everything from a kidnapping to the detonation of a tactical weapon (e.g., chemical, biological, radiological, or nuclear) in their possession. If some prior knowledge about the group’s capability to strike or their intent is known, then there may exist some specific range of severity over which all possibilities are equally likely, but which only includes a portion of the entire range.

### *5.3 Decision Maker Policies*

The CRRA’s MCL mentions nine specific top-level categories. A decision maker within the CRRA process is required to define acceptable levels of risk for shortfalls in each of those nine categories. Assume that a single decision maker – perhaps the Secretary of the Air Force or the Chief of Staff – must bound the limits

of “acceptable risk,” thereby placing his or her influence on the CRRA process. Various weighting schemes exist to accurately capture the decision maker’s risk aversion emphasis areas; see [5] or similar references for a discussion of such methods.

For this example, suppose the decision maker has seen the list of shortfall areas produced in December of 2003, and the shortfall interpretations offered earlier reflect his or her opinions. Assume he or she is least risk averse in MCL areas 8 and 9, slightly risk averse in areas 5, 6, and 7, and most risk averse in areas 1 through 4. Table 5.4 shows the notional “risk-aversion” weights assigned by the decision maker to each of the MCL areas.

Table 5.4 Notional MCL point allocations reflecting decision maker policies.

MCL Item	Assigned Weight
Surveillance & Recon	20
Intelligence	30
Command & Control	19
Communications	13
Force Application	6
Force Projection	6
Protect	6
Prepare & Sustain	0
Create	0

## 5.4 CRRA Risk Scales

The risk scale associated with the CRRA was mentioned in Chapters 1 and 4, and the detailed list of consequence categories and severity factors is shown in Table 5.2; that scale, in its discrete form, features all of the drawbacks discussed in Chapter 4. On the positive side of the ledger, however, Table 5.2 does offer a solution to the problem of “correct” scaling. For the consequence category “Friendly Casualties,” the number of deaths listed is a reflection of how its creator(s) may have intended to weight the severity factors 1-6. This information is summarized in Table 5.5.

Table 5.5 Fatalities associated with severity factors.

Severity Factor	Specified Fatalities	Interpretation	$\log_{10}$
Minor	“Few”	1 - 10	[0, 1]
Modest	“Tens”	10 - 100	(1, 2]
Substantial	“Hundreds”	100 - 1,000	(2, 3]
Major	“Hundreds to Thousands”	1,000 - 10,000	(3, 4]
Extensive	“Thousands to tens of thousands”	10,000 - 100,000	(4, 5]
Catastrophic	“Hundreds of thousands”	100,000 - 1,000,000+	(5, 6+)

Note that the fourth column of Table 5.5 shows that the base 10 logarithm of the fatalities interpretation column corresponds (in essence) to the CRRA’s original scale, only now it is allowed to take on continuous scores over [0,6+]. With some improvements and interpretations, then, a continuous version of the original scale devised by the CRRA planners is a perfectly usable one for SME tabletop discussions; once an appropriate score for a severity, call it  $x$ , has been determined on the

continuous  $[0,6+)$  scale, the weighted severity, call it  $y$ , can be easily recovered by the simple relationship  $y = 10^x$ .

### **5.5 Example SME Data**

While the decision maker is weighting the MCL areas to reflect his or her areas and amounts of greatest risk aversion, suppose that nine teams of SMEs are asked to evaluate the risks associated with any shortfalls they identify in their respective major emphasis areas of the MCL (using the unweighted CRRA risk scale). After much deliberation, the data in Table 5.6 is returned from the process. Note that in some cases, the assessed complexity of the risk scenario causes the SMEs to return only categorical data.

Table 5.6 Notional SME-provided data.

MCL Item	SME-Provided Information				
	$1 - p$	Low	High	Mode	Mean
Surveillance & Recon	0.025	0	Catastrophic	3	N/A
Intelligence	0.002	0	4.67	3.2	N/A
Command & Control	0.0075	Minor	Major	N/A	N/A
Communications	0.01	0	4	2	N/A
Force Application	0.36	Minor	Catastrophic	1.25	N/A
Force Projection	0.03	Minor	Catastrophic	Substantial	N/A
Protect	0.375	Minor	Modest	N/A	N/A
Prepare & Sustain	0.24	Minor	Catastrophic	0	2.2
Create	0.5	Minor	1.875	Minor	N/A

To review the need for the  $1 - p$  column, recall the discussion from Chapter 2 regarding the difference between the risk distribution and the severity distribution. Let the non-negative random variable  $X$  be defined as the severity that could be experienced in a scenario (given that an undesirable outcome occurs), and let  $Y$  be a binary random variable such that

$$Y = \begin{cases} 1 & \text{if outcome occurs} \\ 0 & \text{if outcome does not occur.} \end{cases} \quad (5.1)$$

Now let  $p$  be the probability that no undesirable outcome occurs, so that

$$P\{Y = 0\} = p \quad (5.2)$$

and

$$P\{Y = 1\} = 1 - p. \quad (5.3)$$

The severity distribution,

$$S(x) = 1 - F(x) = P(X > x \mid Y = 1), \quad (5.4)$$

is a conditional probability, where it is assumed that a negative consequence of some severity will occur with certainty ( $p = 0$ ). When  $p \neq 0$ , we have the unconditional risk distribution

$$\begin{aligned} R(x) \equiv P(X > x) &= P(X > x \mid Y = 0) \cdot p \\ &\quad + P(X > x \mid Y = 1) \cdot (1 - p) \end{aligned} \quad (5.5)$$

$$= P(X > x \mid Y = 1) \cdot (1 - p), \quad (5.6)$$

since  $P(X > x \mid Y = 0) = 0$ . Thus the risk distribution includes all of the severity distribution, but also the possibility of no negative consequence whatsoever (i.e., we

are uncertain that *any* event will occur). Now that the center of discussion is *risk* rather than simply *severity*, the expectation must include the constant  $1 - p$ , so

$$E[X \mid Y = 1] = (1 - p) E[X]. \quad (5.7)$$

Using the methodology described in Chapter 4, the distributions reflecting both the CRRA scale  $[0,6+)$  and the “weighted” scale ( $y = 10^x$ ) are determined and recorded in Table 5.7. (For an example of corresponding CRRA and “weighted” distributions plotted on the same axes, see Figure 5.1.) In the case of the weighted Weibull distributions, note that the value of the mode must first be mapped to the weighted scale before  $\theta$  can be determined.

Table 5.7 Notional distributions from SME-provided data.

MCL Item	Distribution (CRR)	Distribution (Weighted)
Surveillance & Recon	Weibull(3.5,3.3)	Weibull(3.5,1101)
Intelligence	Tria(0,4.67,3.2)	Tria(1,46773,1585)
Command & Control	Unif(0,4)	Unif(1,10 <sup>4</sup> )
Communications	Tria(0,4,2)	Tria(1,10 <sup>4</sup> ,100)
Force Application	Weib(2.04,1.74)	Weib(2.04,24.73)
Force Projection	Weib(3.08,2.84)	Weib(3.08,359.1)
Protect	Unif(0,2)	Unif(1,100)
Prepare & Sustain	Exp(0.45)	Exp(0.0063)
Create	Tria(0,1.875,0.5)	Tria(1,75,3.16)



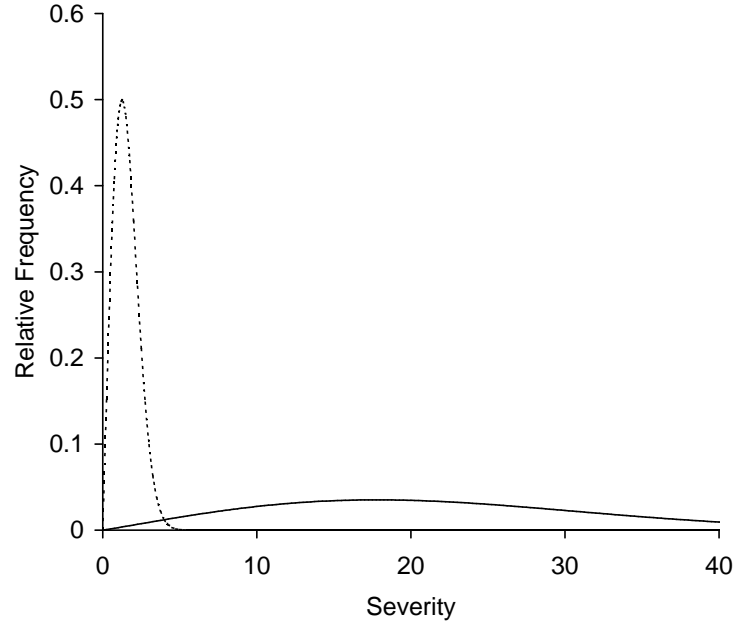


Figure 5.1 Notional CRRA scale and “weighted” scale distributions for Force Application ( $\cdots$  is CRRA, solid is weighted).

### 5.6 *Application of Distortion Functions*

With the SME data channelled into actual distributions and a specified value for  $p$ , the first calculations of shortfall priority, based on expectation, can be completed. For the results shown in Table 5.8, the first column labelled  $(1-p)\mu_0$  reflects the distributions derived from the raw CRRA scale parameters. The second such column reflects distributions with parameters translated to the weighted scale.

Table 5.8 Risk expectations for undistorted distributions, SME-provided data.

	CRRA Scale			Weighted Scale		
MCL Item	$\mu_0$	$1 - p$	$(1 - p)\mu_0$	$\mu_0$	$1 - p$	$(1 - p)\mu_0$
Surv & Recon	2.969	0.025	0.074	990.622	0.025	24.766
Intelligence	2.623	0.002	0.005	16119.670	0.002	32.239
Cmnd & Cntrl	2.000	0.008	0.015	5000.500	0.008	37.504
Communications	2.000	0.010	0.020	3367.000	0.010	33.670
Force App	1.542	0.360	0.555	21.910	0.360	7.887
Force Proj	2.539	0.030	0.076	321.049	0.030	9.631
Protect	1.000	0.375	0.375	50.500	0.375	18.938
Prepare/Sustain	2.200	0.240	0.528	158.730	0.240	38.095
Create	0.792	0.500	0.396	26.387	0.500	13.193

Using the expectation risk measure from the raw CRRA scale data, the shortfalls would receive emphasis in the following order:

1. Force application
2. Prepare and sustain
3. Create the force
4. Protect
5. Force projection
6. Surveillance and reconnaissance
7. Communications
8. Command and control
9. Intelligence.

Using the weighted scale, however, the shortfalls would receive these rankings:

1. Prepare and sustain
2. Command and control
3. Communications
4. Intelligence
5. Surveillance and reconnaissance
6. Protect
7. Create the force
8. Force projection
9. Force application.

### 5.7 Applying Distortion to Risk Distributions

The significant differences between the two lists demonstrate the importance of an accurately weighted scale. If instead of  $y = 10^x$  the weighted scale's creators had used  $y = 5^x$  or  $y = 2^x$ , the order of importance might well again be different. One artificiality of the methodology described in [32] is that the same distortion function/parameter combination is applied to all of the risk distributions to “re-order” the MCL areas. This would not occur in the general case. As the distance between categories grows large (and  $y = 10^x$  is certainly a “large” scale), the ability to apply distortion in this fashion and re-order the risk areas grows weaker. Hence, this thesis proposes applying distortion on a selective, distribution-by-distribution basis, using more limited amounts of distortion according to the will of the decision maker.

To proceed, the decision maker has provided information on his or her priorities, or areas of greatest risk aversion. In Chapter 4 it was stated that the decision maker would likely hold one of two primary objectives: (1) maximize the increase in mean given a specified shift in density, or (2) minimize the amount of density shift required to achieve a specified increase in mean. While either objective could be applied here, we will proceed on the assumption that the decision maker prefers objective (1), and that the weighting given to an area corresponds to a specific shift in density  $R_g$ . Since a weighted scale would certainly be used in a real-world application, it will be used here.

For MCL area 1, surveillance and reconnaissance, a Weib(3.5,1101) distribution was fit, and the decision maker-prescribed amount of distortion was 20%, or  $R_g \approx 1.20$ . At this level, the PH distortion is the most *efficient*, using the definition set forth in Chapter 4. Setting  $a = 0.735$  results in  $\mu_g = 1081.7$ , and  $(1-p)\mu_g = 27.043$ . We continue in this fashion for all of the risk distributions, applying distortions based on the recommendations of Table 4.11. Table 5.9 summarizes the results of selectively distorting as per the *a priori* instructions of the decision maker. In the table, column

$(1 - p)\mu_0$  is the original undistorted risk expectation,  $R_g$  is the decision maker's desired amount of distortion, "Combination" is the selected distortion function and its associated parameter value, and  $(1 - p)\mu_g$  is the risk measure after distortion is applied.

Table 5.9 Results for application of selected distortion functions to notional risk distributions, weighted scale.

MCL Item	Distribution	$(1 - p)\mu_0$	$R_g$	Combination	$(1 - p)\mu_g$
Surv & Recon	Weib(3.5,1101)	24.766	1.20	PH, $a = 0.735$	27.043
Intelligence	Tria(1,46773,1585)	32.239	1.30	PH, $a = 0.62$	42.641
Cmnd & Cntrl	Unif(1,10 <sup>4</sup> )	37.504	1.19	EX, $c = 1.3$	42.264
Communications	Tria(1,10 <sup>4</sup> ,100)	33.670	1.13	EX, $c = 1.9$	37.189
Force App	Weib(2.04,24.73)	7.887	1.06	PH, $a = 0.915$	8.238
Force Proj	Weib(3.08,359.1)	9.631	1.06	PH, $a = 0.915$	9.913
Protect	Unif(1,100)	18.938	1.06	DP, $b = 1.09$	19.737
Prepare/Sustain	Exp(0.0063)	38.095	1.0	N/A	38.095
Create	Tria(1,75,3.16)	13.193	1.0	N/A	13.193

Using the weighted scale, Table 5.10 provides a side-by-side comparison of pre- and post-distortion priorities. Compared to the original weighted, undistorted risk measure, intelligence has vaulted to the top of the rankings through selectively applied distortion.

Table 5.10 Comparison of pre- and post-distortion priorities using weighted scale, SME-provided data.

MCL Item	Pre-distortion	Post-distortion
Surveillance & Recon	5	5
Intelligence	4	1
Command & Control	2	2
Communications	3	4
Force Application	9	9
Force Projection	8	8
Protect	6	6
Prepare & Sustain	1	3
Create	7	7

## 5.8 *Selection of Systems to Counter Capability Shortfalls*

One of the objectives of the CRRA process is to propose risk mitigation measures, making connections between acquisitions programs and required capabilities. In [32], Woodward addressed this goal by using an integer programming (IP) methodology and expectation risk measures; the key difference is that he uniformly applies the DP distortion function with a constant parameter  $b$  value across all areas of the MCL. A comparison between the results in [32] and those here will be presented after a review of Woodward's [32] procedure and application of the current results to his methodology.

First, a small set of notional acquisitions programs are presented, and the "percent shortfall mitigation" each program addressed in all nine areas of the MCL is provided. The notional programs mentioned in [32] are

1. an intelligence database with global access capability;
2. a new heads-up display (HUD) to improve situational awareness in fighter aircraft;
3. a stand-off missile system with the capability to track and destroy moving targets;
4. an unmanned aerial vehicle (UAV) designed for battle damage assessment (BDA);
5. advanced equipment for detecting the presence of chemical, biological, radiological, nuclear, and high-yield explosives (CBRNE); and
6. additional air-to-air refueling aircraft ("tankers").

Table 5.11 reflects data *similar* to that found in [32]. To interpret the table, the fighter HUD, for example, reduces the shortfall in surveillance and reconnaissance by 19%.

Table 5.11 Notional acquisitions programs and percent shortfall mitigation per MCL top-level area [32].

MCL Item	Intel Database (1)	Fighter HUD (2)	Standoff Missile (3)	BDA UAV (4)	CBRNE Detection (5)	Refuelers (6)
(1) Surv & Recon	0	0.19	0	0.26	0.26	0
(2) Intelligence	0.46	0.21	0	0.12	0	0
(3) Cmnd & Cntrl	0.34	0.19	0	0.05	0	0.23
(4) Communications	0.14	0.42	0	0.36	0	0.05
(5) Force App	0.10	0.21	0.92	0.30	0	0.10
(6) Force Proj	0	0	0.54	0	0	0.11
(7) Protect	0.16	0	0	0	0.25	0
(8) Prepare/Sustain	0.19	0	0	0	0.31	0.48
(9) Create	0	0	0	0	0.33	0.36
Cost, $k$	7	7	10	8	8	9



The last line of Table 5.11 reflects a cost associated with the purchase of the *complete* system. The assumptions for the upcoming discussion are that not all of the systems can be afforded (suppose a budget of 25 units), and that fractions of each system *can* be purchased. The last point is a change from [32], which used an integer program (“all or nothing”) in solving the problem; this change is justifiable due to the recent USAF/DoD emphasis on “spiral development,” where very basic capabilities are purchased as they become available and improvements are made over the life of the system.

Let  $m_{ij}$  denote the percent shortfall mitigation to MCL area  $i$  derived from system  $j$ , where  $i = 1, 2, \dots, 9$  and  $j = 1, 2, \dots, 6$ . If all areas of the MCL were viewed as equally important, then the overall contribution of any complete system  $j$  to the USAF,  $C_j$ , would be the sum of its percent shortfall mitigation measures from Table 5.11, i.e.,  $C_j = \sum_{i=1}^9 m_{ij}$ . To optimize the solution in that case, we would implement a linear program for which the objective function coefficients are the  $C_j$ ’s and the objective function variables are the “amounts” of each system to be purchased,  $x_j$ . If the cost of any *complete* system  $j$  is  $k_j$ , then the only acquisition constraints are

$$\sum_{j=1}^6 k_j x_j \leq 25 \quad (5.8)$$

and

$$0 \leq x_j \leq 1, \quad (5.9)$$

since the total costs of the procured systems cannot exceed 25 units and either none, some, or all of a complete system can be purchased.

Of course, a main point of this thesis has been that not all risks have equal consequences, and that the risk distributions can be summarized and distorted. It seems reasonable, then, to weight each  $m_{ij}$  with the risk expectations which accompany each MCL area  $i$ , and these will be denoted as  $S_i$ . Including these weights, the

entire LP becomes

$$\begin{aligned}
& \text{Maximize} && \sum_{i=1}^9 \sum_{j=1}^6 S_i m_{ij} x_j \\
& \text{subject to} && \sum_{j=1}^6 k_j x_j \leq 25 \\
& && 0 \leq x_j \leq 1, \quad j = 1, 2, \dots, 6,
\end{aligned} \tag{5.10}$$

which is directly comparable to the mathematical program of [32].

Both the “weighted with undistorted expectation” and “weighted with distorted expectation” solutions to the example problem were solved, meaning that the LPs were identical except for the values of  $S_i$ . The resulting “purchases” recommended by each of the two LPs are shown in Table 5.12, along with the unweighted solution using the  $C_j$ ’s as the objective function coefficients.

Table 5.12 Notional acquisitions programs purchase recommendations, by type of expectation applied.

Expectation Applied	Intel Database	Fighter HUD	Standoff Missile	BDA UAV	CBRNE Detection	Refuelers
None	1	1	0.2	0	0	1
Weighted, Undistorted	1	1	0	0	0.25	1
Weighted, Distorted	1	1	0	0.25	0	1

While all of the different purchase plans recommend the intel database, fighter HUD, and refuelers be bought first, the fractional expenditures change over all three: when no expectation is applied, initial expenditure in the standoff missile system is recommended; when weighted, undistorted expectation is applied, some investment

in CBRNE detection is part of the solution; and using weighted, distorted expectation, remaining funds should be directed toward the BDA UAV.

For all three of the solutions, a *decrease* of more than two units in the budget will result in a change to the optimal mix of systems. A budget *increase* of eight units for the unweighted solution – or six for both the weighted, distorted and weighted, undistorted solutions – can be absorbed before a change is observed in the mix of systems. Among the objective coefficient ranges, the smallest changes that will result in a new solution to the LP are included in Table 5.13.

Table 5.13 Notable objective coefficient sensitivities for system mix LP with various expectations applied.

Expectation Applied	Variable	Current Coefficient	Allowable Increase	Allowable Decrease
None	$x_3$	1.46	0.0178	0.0225
None	$x_5$	1.15	0.018	$\infty$
None	$x_6$	1.33	$\infty$	0.016
Weighted, Undistorted	$x_4$	240.033	5.998	$\infty$
Weighted, Undistorted	$x_5$	246.031	35.51	5.998
Weighted, Distorted	$x_4$	271.087	21.149	17.929
Weighted, Distorted	$x_5$	253.157	17.929	$\infty$
Weighted, Distorted	$x_6$	328.765	$\infty$	23.793

## 5.9 Flowchart of Proposed Methodology

To summarize the entire process described by the thesis and aid the reader, a flowchart of the proposed methodology is included in Figure 5.2.

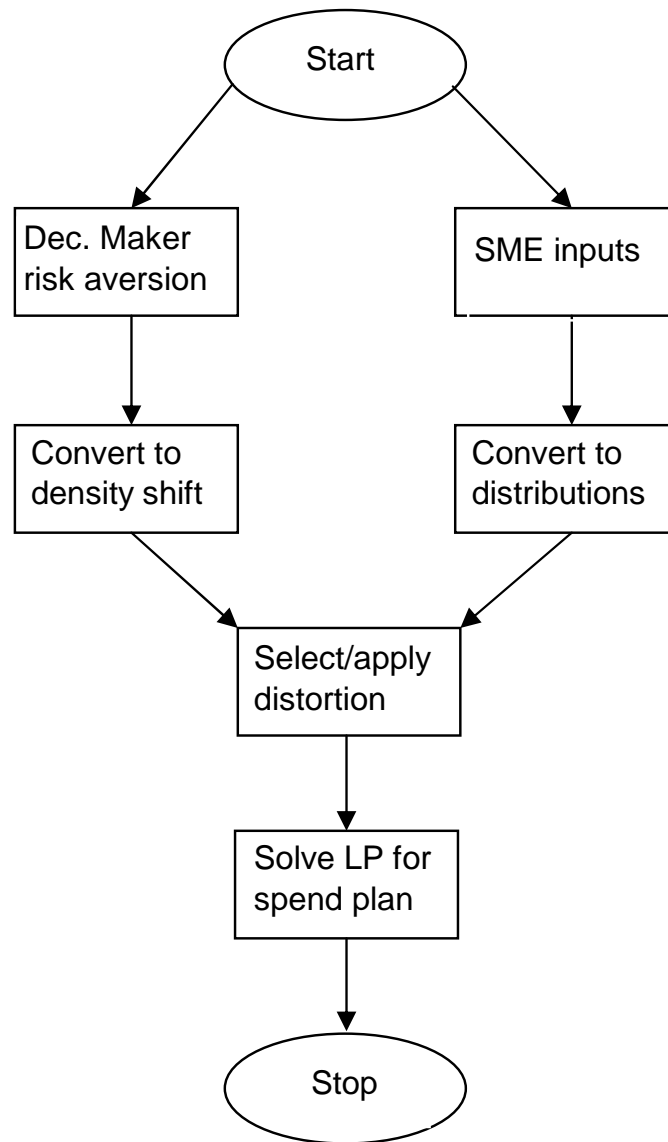


Figure 5.2 Flowchart of proposed methodology.

### 5.10 Comparison with Previous Results

To form a complete picture, a comparison with Woodward's [32] results is in order. To begin, Woodward used the same systems but slightly different values for shortfall mitigations and costs; his exact values are reproduced in Table 5.14.

Table 5.14 Exact notional acquisitions programs and percent shortfall mitigation per MCL top-level area, as listed in [32].

MCL Item	Intel Database (1)	Fighter HUD (2)	Standoff Missile (3)	BDA UAV (4)	CBRNE Detection (5)	Refuelers (6)
(1) Surv & Recon	0	0.19	0	0.26	0.26	0
(2) Intelligence	0.46	0.21	0	0.12	0.68	0
(3) Cmnd & Cntrl	0.34	0.19	0	0	0	0.23
(4) Communications	0.14	0.42	0	0.36	0	0.05
(5) Force App	0.10	0.21	0.92	0.30	0	0.10
(6) Force Proj	0	0.16	0.54	0	0	0.11
(7) Protect	0.16	0	0	0	0.25	0
(8) Prepare/Sustain	0.19	0	0	0	0.31	0.48
(9) Create	0.1	0.05	0	0	0.33	0.36
Cost, $k$	7	7	10	9	6	9

Additionally, Woodward [32] used an integer programming (IP) approach to the problem (meaning portions of a system could not be acquired), a total budget of 23 units (instead of 25), and his own derived distributions on a  $\log_2$  severity index (meaning severity doubles for each categorical step increase). Since all of his chosen distributions are either exponential or Weibull, Table 4.11 will always recommend the PH distortion when using the methodology advocated here, but the value of distortion parameter  $a$  will differ for each distribution based on the decision maker's inputs.

With these modifications, the selective application of distortion can be directly compared to Woodward's "blanket" application of only the DP distortion using  $b = 5$  and  $b = 10$  (sequentially) to all of the risk distributions at once [32]. Table 5.15 shows the distributions Woodward used for each risk area of the MCL, associated data, and the undistorted risk measures, while Table 5.16 provides the expectations and rankings for each area after the application of different distortion functions, with both "blanket" and selective methods examined.

Table 5.15 Distributions assigned to MCL items by [32], associated data, and undistorted expectation risk measures and rankings.

MCL Item	Distribution	Distribution Mean ( $\mu$ )	$1 - p$	$(1 - p)\mu_0$ (Rank)
Surveillance & Recon	Weib(1.453,28.99)	26.27	0.5375	14.1 (1)
Intelligence	Exp(0.102)	9.77	0.48	4.7 (9)
Command & Control	Exp(0.042)	23.64	0.43	10.17 (5)
Communications	Weib(1.136,23.53)	22.48	0.5275	11.8 (3)
Force Application	Weib(1.639,28.49)	25.49	0.5225	13.3 (2)
Force Projection	Weib(0.98,15.55)	15.69	0.4325	6.8 (8)
Protect	Exp(0.040)	24.75	0.3475	8.6 (7)
Prepare & Sustain	Weib(1.702,30.58)	27.28	0.385	10.5 (4)
Create	Exp(0.034)	29.85	0.34	10.15 (6)

Table 5.16 Distorted expectation risk measures and rankings for MCL item distributions [32].

MCL Item	Woodward [32]		Proposed Methodology	
	DP, $b = 5$ Mean (Rank)	DP, $b = 10$ Mean (Rank)	Selective Distortion Applied	Mean (Rank)
Surveillance & Recon	26.7 (1)	32.1 (1)	PH, $a = 0.735$	17.5 (1)
Intelligence	10.7 (9)	13.7 (9)	PH, $a = 0.62$	7.6 (8)
Command & Control	23.2 (4)	29.8 (3)	PH, $a = 0.75$	13.6 (4)
Communications	25.3 (2)	31.7 (2)	PH, $a = 0.82$	14.12 (2)
Force Application	23.9 (3)	28.1 (5)	PH, $a = 0.915$	14.06 (3)
Force Projection	15.7 (8)	20.2 (8)	PH, $a = 0.915$	7.4 (9)
Protect	19.6 (6)	25.2 (6)	PH, $a = 0.915$	9.4 (7)
Prepare & Sustain	18.5 (7)	21.7 (7)	N/A	10.5 (5)
Create	23.2 (5)	29.7 (4)	N/A	10.15 (6)



Table 5.16 shows that all of the methods of applying distortion have the ability to change the rankings of the MCL items, but it does not state whether or not the subset of mitigating systems recommended for purchase is changed based on the distortion applied. That information is included in Table 5.17, where it is observed that undistorted expectation, the “blanket” application of the DP distortion with  $b = 5$ , the “blanket” DP approach with  $b = 10$ , and the selective distortion method all result in the same purchase plan.

Table 5.17 Comparison of notional acquisitions programs purchase recommendations, by type of expectation applied, using selective distortion and “blanket” distortion methodologies (1 is “buy,” 0 is “do not buy”).

Expectation Applied	Total Cost	Intel Database	Fighter HUD	Standoff Missile	BDA UAV	CBRNE Detect	Refuelers
Undist. expectation	23	0	1	1	0	1	0
All DP, $b = 5$ [32]	23	0	1	1	0	1	0
All DP, $b = 10$ [32]	23	0	1	1	0	1	0
Selective distortion	23	0	1	1	0	1	0

Although the end solution to the problem is the same in all cases when the situation is governed by Woodward’s “rules,” there are at least two distinct advantages to the current methodology as compared to [32]. First, in the proposed methodology there is more freedom in the selection and shape of the probability distributions used to model risk scenarios. While the method of selecting distributions presented here is rudimentary, the data requested from the SMEs is similarly straightforward (specifically the range and mode of the expected consequence, and perhaps the mean). Second, the proposed methodology attempts to apply distortion for the purpose of

imposing the will of the decision maker on the SME-provided risk distributions, rather than applying distortion en masse and observing the effects. While it may seem on the surface that the application of the same distortion function/parameter combination to all distributions is more objective, in reality the effect of a distortion on a given distribution is predictable to some degree. For instance, recall in the case of the exponential distribution that an increase in distortion generally results in an increase in distortion efficiency, while other distributions experience the opposite effect; hence, the risk scenarios described by the exponential should be expected to move up in the rankings as more distortion is applied.

### **5.11 *Summary***

This chapter presented real-world background material and analysis that formed the basis of a notional example. A method for turning decision maker prioritization policies into specific distortion function and parameter combinations was examined, and these pairings were applied to probability distributions based on the notional analysis of fictitious SMEs. Using a mathematical programming methodology first described in [32], the selectively distorted risk measures were used as weights in a linear programming problem to determine an optimal subset of risk mitigating systems given a limited budget. To complete the chapter, the methodology of this chapter was applied directly to the exact scenario and distributions from [32] and the results compared.

## 6. Conclusions and Future Research

### 6.1 *Summary of Contributions*

In examining and quantifying risk, the researcher in the field must make a clear distinction between risk and severity. Risk includes both the severity in the event of occurrence and the probability that no event occurs at all, while severity assumes that some negative occurrence will happen with certainty. Many different probability distributions are used in the disciplines of health, finance, and reliability, where measuring risk is of high importance. Regardless of the distribution in use, however, mathematical expectation (either by itself or combined with another measure) remains the dominant risk measure across all fields. Distorted expectation can be considered another way of measuring risk. Distortion functions are a relatively new area of research in financial risk analysis; they produce a predictable, positive shift of density in a distribution. However, very little information has been published regarding which situations signal the appropriateness of one distortion function over another.

A well-defined risk scale is essential to risk analysis. In most real-world cases, the risk scale should be continuous, infinite, and appropriately scaled. The idea that some distributions are defined over a finite range and others over an infinite range is an important distinction to make in selecting a risk distribution to model a risk scenario; the amount and directions of skewness a distribution can take on are also discriminating factors. A simple decision tree can be formulated to pinpoint an appropriate severity distribution if the potential distributions have unique characteristics of range and shape, as is the case with the exponential, Weibull, triangular, and uniform distributions. For those four distributions, distribution parameters can

be determined with just an assessment of the mode (or mean in the case of the exponential distribution).

In examining the effects of distortion, the amount of density shift should be considered when a change in expectation is desired. This is because the more density that is shifted, the more the risk distribution differs from that prescribed by the subject matter experts. Therefore, the decision maker should either shift density as little as possible while achieving a maximum amount of gain in expectation, or increase expectation as much as possible given a set amount of density shift.

Multi-parameter distortions such as the gamma-beta have the capacity to change a distribution in more than one way simultaneously (by not focusing on just a single tail of the distribution). The primary difficulty involved with using the multi-parameter distortions is the intractability of the resulting integrals when the expectations are pursued using the survivor function (even some of the single-parameter distortions presented apparently intractable integrals).

The single-parameter proportional hazard, dual power, and exponential distortions all shift density in different ways, with the PH typically having the greatest effect on the right tail of the distribution and the DP generally having the least. Depending on the distribution and how much change in expectation is required or requested by the decision maker, different distortion functions may therefore be more *efficient* over different amounts of change in mean, where efficiency is defined as the change in expectation over the shift in density.

Some references in the past have applied the same distortion function/parameter combination to all of the risk distributions at once. This thesis holds that a superior method is for the decision maker to pick his areas of emphasis and apply distortion on a case-by-case basis without prior knowledge of the expectations. By using just the single-parameter distortion functions within the *coherent* parametric limits, a decision maker has the power to shift density within a risk distribution by a precise amount and put a personal stamp on the resulting measures.

## 6.2 *Future Research Considerations*

Since this thesis appears to be the first attempt at providing recommendations for selecting distortion function/parameter and distribution combinations, there are many options available for future study. The following list provides only a few of them.

1. An efficient technique to compute the expectation of the gamma-beta distorted random variable (and the other multi-parameter distortions) must be found. In particular, the beta distortion is referenced in the literature, but the inability to calculate the expectation made it difficult to comment about it within the thesis.
2. Measures other than  $R_g$  should be considered; perhaps a measure of the movement of the mode along the severity ( $x$ ) axis would be preferable. The limitations of  $R_g$  in terms of the “region of sensitivity” weaken it as a measure.
3. Ways to quantify/assess decision maker hesitancy in deviating from the recommendations of his SMEs should be examined. While it seems that a 30% shift in density is considerable, no simple method of explaining this concept to the decision maker has been presented here.
4. A formal study of the relationship between the skewness of a distribution (perhaps using Pearson’s skewness coefficient) and either the percent change in expectation or the  $R_g$  should be undertaken. There is some correlation between Pearson’s coefficient and the normalized mean; it appears that certain distributions (the exponential is one case) provide more change in mean than others when distortion is applied. This is easily observable in Table 4.10 along the exponential distribution data.
5. The application of distortion functions to other primary distributions across various fields should be examined, not just the four distributions listed in this thesis. One particular limitation of the selected distributions is that no

left-skewed, infinite-tailed distribution is included (meaning left-skewed only over the *defined* range of severity). Also, the potential of interactions between distribution parameters and distortion parameters should be considered.

6. If the variance of a distorted distribution can be easily computed, then risk-value models discussed in [23] may have some utility as a way to summarize a distribution *after* distortion has been applied, rather than simply being a competitor to distorted expectation *beforehand*. More generally, further research regarding the effects of distortion on variance may have a major effect on what constitutes the “best” distortion function to apply to a given risk scenario.

### **6.3 Final Observations**

This thesis has only scratched the surface of forming a rule base for the selection of a distortion function. It does seem clear that in SME-examined risk scenarios, unbounded distortion does not well-serve the decision maker or the institution which hired the SMEs, as more distortion further discounts SME opinions. Additionally, applying the same distortion function/parameter combination to all risk scenarios, which seems on the surface to be a way of using distortion without “looking at the data” first, will in fact have a predictable effect on the risk scenarios. A superior method is believed to be using distortion selectively to push individual risk scenarios ahead of others on a case-by case basis in accordance with the wishes of the decision maker.

Given the budgetary importance of the CRRA process, distortion should be given considerable attention as a way to objectively rank risk scenarios while still allowing the “gut instincts” of senior-level decision makers to influence the acquisitions process.

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## Appendix A. MatLab® Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following code is used in the creation of Table 4.10.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% call_others_efef.m
% Author: Maj Edwin J. Offutt
% This m-file is used to call other m-files in the construction of Table
% 4.10 in the thesis document. Other m-files called by this one are
% included below.

clear all; clc; table = []; table = zeros(16,9);

% 1
[op] = efexpoph(0.9,0.198042,3.5,10^-5,10^6); table(1:4,1) = op;
% 2
[op] = efexpoph(0.75,0.198042,3.5,10^-5,10^6); table(1:4,2) = op;
% 3
[op] = efexpoph(0.6,0.198042,3.5,10^-5,10^6); table(1:4,3) = op;
% 4
[op] = efexpodp(1.1,0.198042,3.5,10^-5,10^6); table(1:4,4) = op;
% 5
[op] = efexpodp(1.3,0.198042,3.5,10^-5,10^6); table(1:4,5) = op;
% 6
[op] = efexpodp(1.5,0.198042,3.5,10^-5,10^6); table(1:4,6) = op;
% 7
[op] = efexpoex(3.6,0.198042,3.5,10^-5,10^6); table(1:4,7) = op;
% 8
[op] = efexpoex(2.2,0.198042,3.5,10^-5,10^6); table(1:4,8) = op;
% 9
[op] = efexpoex(0.8,0.198042,3.5,10^-5,10^6); table(1:4,9) = op;
% 10
[op] = efweibph(0.9,1.66666667,2,2,10^-5,10^6); table(5:8,1) = op;
% 11
[op] = efweibph(0.75,1.66666667,2,2,10^-5,10^6); table(5:8,2) =
op;
% 12
[op] = efweibph(0.6,1.66666667,2,2,10^-5,10^6); table(5:8,3) = op;
% 13
[op] = efweibdp(1.1,1.66666667,2,2,10^-5,10^6); table(5:8,4) = op;
% 14
```

```

[op] = efweibdp(1.3,1.66666667,2,2,10^-5,10^6); table(5:8,5) = op;
% 15
[op] = efweibdp(1.5,1.66666667,2,2,10^-5,10^6); table(5:8,6) = op;
% 16
[op] = efweibex(3.6,1.66666667,2,2,10^-5,10^6); table(5:8,7) = op;
% 17
[op] = efweibex(2.2,1.66666667,2,2,10^-5,10^6); table(5:8,8) = op;
% 18
[op] = efweibex(0.8,1.66666667,2,2,10^-5,10^6); table(5:8,9) = op;
% 19
[op] = eftriaph(0.9,4,1,7,4,10^-5); table(9:12,1) = op;
% 20
[op] = eftriaph(0.75,4,1,7,4,10^-5); table(9:12,2) = op;
% 21
[op] = eftriaph(0.6,4,1,7,4,10^-5); table(9:12,3) = op;
% 22
[op] = eftriadp(1.1,4,1,7,4,10^-5); table(9:12,4) = op;
% 23
[op] = eftriadp(1.3,4,1,7,4,10^-5); table(9:12,5) = op;
% 24
[op] = eftriadp(1.5,4,1,7,4,10^-5); table(9:12,6) = op;
% 25
[op] = eftriaex(3.6,4,1,7,4,10^-5); table(9:12,7) = op;
% 26
[op] = eftriaex(2.2,4,1,7,4,10^-5); table(9:12,8) = op;
% 27
[op] = eftriaex(0.8,4,1,7,4,10^-5); table(9:12,9) = op;
% 28
[op] = efunifph(0.9,4,1,7,10^-5); table(13:16,1) = op;
% 29
[op] = efunifph(0.75,4,1,7,10^-5); table(13:16,2) = op;
% 30
[op] = efunifph(0.6,4,1,7,10^-5); table(13:16,3) = op;
% 31
[op] = efunifdp(1.1,4,1,7,10^-5); table(13:16,4) = op;
% 32
[op] = efunifdp(1.3,4,1,7,10^-5); table(13:16,5) = op;
% 33
[op] = efunifdp(1.5,4,1,7,10^-5); table(13:16,6) = op;
% 34
[op] = efunifex(3.6,4,1,7,10^-5); table(13:16,7) = op;
% 35
[op] = efunifex(2.2,4,1,7,10^-5); table(13:16,8) = op;

```

```

% 36
[op] = efunifex(0.8,4,1,7,10^-5); table(13:16,9) = op;

table

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following 12 programs (one for each combination of the
% single-parameter distortion functions and the featured
% distributions) can be used to compute the effectiveness and
% efficiency values shown throughout Chapters 4 and 5.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 1. Exponential distribution and DP distortion

function [op] = efexpodp(b,mval,lda,ep,N)
% efexpodp(b,mval,lda,ep,N)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the DP distortion
% as applied to the exponential distribution. The user enters values for b
% (the distortion parameter), mval (the undistorted median), lda (lambda,
% the exponential parameter), ep (the precision in the numerical integration,
% typically 10^-14), and N (the upper limit of the range of integration,
% typically 10^5 or more).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_OS = []; func_OS = [];

% ep = 10^-6;

S=inline(['exp(-' num2str(lda) '*x)']);
Seval=feval(S,mval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);

```

```

% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (1-t_01(k1))^(b-1);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_0S)
    func_0S(k2) = (1-t_0S(k2))^(b-1);
end
geval = K*trapz(t_0S,func_0S);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['1-((1-(exp(-' num2str(lda) '.*x))).^(' num2str(b) '.*1))']);
xpec = quad(gS,0,N,ep);
mu0 = quad(S,0,N,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 2. Exponential distribution and EX distortion

function [op] =
efexpoex(c,mval,lda,ep,N)
% efexpoex(c,mval,lda,ep,N)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the EX distortion
% as applied to the exponential distribution. The user enters values for c
% (the distortion parameter), mval (the undistorted median), lda (lambda,
% the exponential parameter), ep (the precision in the numerical integration,
% typically 10^-14), and N (the upper limit of the range of integration,
% typically 10^5 or more).

```

```

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_OS = []; func_OS = [];

% ep = 10^-6;

S=inline(['exp(-' num2str(lda) '*x)']);
Seval=feval(S,mval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = exp(-t_01(k1)/c);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = exp(-t_OS(k2)/c);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['(1-(exp(-(exp(-' num2str(lda) '*x))./'
    ' num2str(c) '))))/(1-(exp(-(1./' num2str(c) '))))')']);
xpec = quad(gS,0,N,ep);
mu0 = quad(S,0,N,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

```

op

%%%

% 3. Exponential distribution and PH distortion

```
function [op] = efexpoph(a,mval,lda,ep,N)
% efexpoph(a,mval,lda,ep,N)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the PH distortion
% as applied to the exponential distribution. The user enters values for a
% (the distortion parameter), mval (the undistorted median), lda (lambda,
% the exponential parameter), ep (the precision in the numerical integration,
% typically 10-14), and N (the upper limit of the range of integration,
% typically 105 or more).
```

```
format compact format long
```

```
% create array
```

```
op=[]; op=zeros(4, 1);
```

```
t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);
```

```
t_0S = []; func_0S = [];
```

```
% ep = 10-6;
```

```
S=inline(['exp(-' num2str(lda) '*x)']);
Seval=feval(S,mval);
Seval = double(Seval);
t_0S = 0.000002:0.000002:Seval;
func_0S = 1:length(t_0S);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (t_01(k1))(a-1);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
```

```

% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_0S)
    func_0S(k2) = (t_0S(k2))^(a-1);
end
geval = K*trapz(t_0S,func_0S);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['(exp(-' num2str(lda) '.*x)).^(' num2str(a) '.*1)']);
xpec = quad(gS,0,N,ep);
mu0 = quad(S,0,N,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 4. Triangular distribution and DP distortion

function [op] = eftriadp(b,mval,a1,b1,c1,ep)
% eftriadp(b,mval,a1,b1,c1,ep)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the DP distortion
% as applied to the triangular distribution. The user enters values for b
% (the distortion parameter), mval (the undistorted median), a1 (the lower
% limit of the triangular distribution), b1 (the upper limit), c1 (the mode)
% and ep (the precision in the numerical integration, typically 10^-14).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

```

```

t_OS = []; func_OS = [];

% ep = 10^-6;

S1=inline(['1-(((x-' num2str(a1) ').^2)/(((' num2str(b1) '-
' num2str(a1) ')*((' num2str(c1) '-' num2str(a1) ')))')]);
S2=inline(['(((' num2str(b1) '-x).^2)/(((' num2str(b1) '-
' num2str(a1) ')*((' num2str(b1) '-' num2str(c1) ')))')]);
if mval <= c1
    Seval=feval(S1,mval);
else
    Seval=feval(S2,mval);
end
Seval = double(Seval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);
% figure integral value over [0,1]
for k1 = 1:length(t_OS)
    func_01(k1) = (1-t_OS(k1))^(b-1);
end
Q = trapz(t_OS,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = (1-t_OS(k2))^(b-1);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS1 = inline(['1-((1-( (1-(((x-' num2str(a1) ').^2)/(((' num2str(b1) '-' num2str(a1) '
' num2str(c1) '-' num2str(a1) '))) )).^(' num2str(b) '.*1))')]);
gS2 = inline(['1-((1-( ((((' num2str(b1) '-x).^2)/(((' num2str(b1) '-' num2str(a1) ')*(
' num2str(b1) '-' num2str(c1) '))) )).^(' num2str(b) '.*1))')]);
xpec1 = quad(gS1,a1,c1,ep);
xpec2 = quad(gS2,c1+ep,b1,ep);
xpec = a1 + xpec1 + xpec2;
mu01 = quad(S1,a1,c1,ep);
mu02 = quad(S2,c1+ep,b1,ep);
mu0 = a1 + mu01 + mu02;
Z = xpec/mu0;

```



```

        op(1) = xpec;
        op(2) = rat;
        op(3) = Z;
        op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 5. Triangular distribution and EX distortion

function [op] =
eftriaex(c,mval,a1,b1,c1,ep)
% eftriaex(c,mval,a1,b1,c1,ep)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the EX distortion
% as applied to the triangular distribution. The user enters values for c
% (the distortion parameter), mval (the undistorted median), a1 (the lower
% limit of the triangular distribution), b1 (the upper limit), c1 (the mode)
% and ep (the precision in the numerical integration, typically 10^-14).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_0S = []; func_0S = [];

% ep = 10^-6;

S1=inline(['1-(((x-' num2str(a1) ')'.^2)/(( ' num2str(b1) '-
' num2str(a1) ')*(' num2str(c1) '-' num2str(a1) ')))']);
S2=inline(['((' num2str(b1) '-x).^2)/(( ' num2str(b1) '-
' num2str(a1) ')*(' num2str(b1) '-' num2str(c1) ')))']);
if mval <= c1
    Seval=feval(S1,mval);
else
    Seval=feval(S2,mval);

```

```

end
Seval = double(Seval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);
% figure integral value over [0,1]
for k1 = 1:length(t_O1)
    func_O1(k1) = exp(-t_O1(k1)/c);
end
Q = trapz(t_O1,func_O1);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = exp(-t_OS(k2)/c);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS1 = inline(['(1-(exp(-( (1-((x-' num2str(a1) ').^2)/(( ' num2str(b1) '-
    ' num2str(a1) ').*( ' num2str(c1) '- ' num2str(a1) ')))) )/' num2str(c) '
    ))))/(1-(exp(-(1./ ' num2str(c) '))))')]);
gS2 = inline(['(1-(exp(-( ( (( ' num2str(b1) '-x).^2)/(( ' num2str(b1) '-
    ' num2str(a1) ').*( ' num2str(b1) '- ' num2str(c1) ')))) )/' num2str(c) '
    ))))/(1-(exp(-(1./ ' num2str(c) '))))')]);
xpec1 = quad(gS1,a1,c1,ep);
xpec2 = quad(gS2,c1+ep,b1,ep);
xpec = a1 + xpec1 + xpec2;
mu01 = quad(S1,a1,c1,ep);
mu02 = quad(S2,c1+ep,b1,ep);
mu0 = a1 + mu01 + mu02;
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

```

op

%%%

```

% 6. Triangular distribution and PH distortion

function [op] = eftriaph(a,mval,a1,b1,c1,ep)
% eftriaph(a,mval,a1,b1,c1,ep)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the PH distortion
% as applied to the triangular distribution. The user enters values for a
% (the distortion parameter), mval (the undistorted median), a1 (the lower
% limit of the triangular distribution), b1 (the upper limit), c1 (the mode)
% and ep (the precision in the numerical integration, typically 10^-14).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_0S = []; func_0S = [];

% ep = 10^-6;

S1=inline(['1-(((x-' num2str(a1) ').^2)/(( ' num2str(b1) '-
' num2str(a1) ')*(' num2str(c1) '- ' num2str(a1) ')))']);
S2=inline(['((' num2str(b1) '-x).^2)/(( ' num2str(b1) '-
' num2str(a1) ')*(' num2str(b1) '- ' num2str(c1) '))']);
if mval <= c1
    Seval=feval(S1,mval);
else
    Seval=feval(S2,mval);
end
Seval = double(Seval);
Seval = double(Seval);
t_0S = 0.000002:0.000002:Seval;
func_0S = 1:length(t_0S);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (t_01(k1))^(a-1);
end
Q = trapz(t_01,func_01);

```

```

Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = (t_OS(k2))^(a-1);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS1 = inline(['(1-((x-' num2str(a1) ')^2)/(( ' num2str(b1) '-' num2str(a1) '
    ).*( ' num2str(c1) '-' num2str(a1) ')))).^(' num2str(a) '.*1)']);
gS2 = inline(['((( ' num2str(b1) '-x).^2)/(( ' num2str(b1) '-' num2str(a1) '
    ).*( ' num2str(b1) '-' num2str(c1) ')))).^(' num2str(a) '.*1)']);
xpec1 = quad(gS1,a1,c1,ep);
xpec2 = quad(gS2,c1+ep,b1,ep);
xpec = a1 + xpec1 + xpec2;
mu01 = quad(S1,a1,c1,ep);
mu02 = quad(S2,c1+ep,b1,ep);
mu0 = a1 + mu01 + mu02;
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 7. Uniform distribution and DP distortion

function [op] =
efunifdp(b,mval,t1,t2,ep)
% efunifdp(b,mval,t1,t2,ep)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the DP distortion
% as applied to the uniform distribution. The user enters values for b
% (the distortion parameter), mval (the undistorted median), t1 (the lower
% limit of the uniform distribution), t2 (the upper limit),
% and ep (the precision in the numerical integration, typically 10^-14).

```

```

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_OS = []; func_OS = [];

% ep = 10^-6;

S=inline(['1 - ((x-' num2str(t1) ')/(' num2str(t2) '-' num2str(t1) '))']);
Seval=feval(S,mval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (1-t_01(k1))^(b-1);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = (1-t_OS(k2))^(b-1);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['1-((1-( 1 - ((x-' num2str(t1) ')/(' num2str(t2) '
    '-' num2str(t1) ')) )).^(' num2str(b) '.*1))']);
xpec = t1 + quad(gS,t1,t2,ep);
mu0 = t1 + quad(S,t1,t2,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

```

op

%%%

% 8. Uniform distribution and EX distortion

```
function [op] =  
efunifex(c,mval,t1,t2,ep)  
% efunifex(c,mval,t1,t2,ep)  
% Author: Maj Edwin J. Offutt  
% This m-file calculates efficiency and effectiveness of the EX distortion  
% as applied to the uniform distribution. The user enters values for c  
% (the distortion parameter), mval (the undistorted median), t1 (the lower  
% limit of the uniform distribution), t2 (the upper limit),  
% and ep (the precision in the numerical integration, typically 10^-14).
```

```
format compact format long
```

```
% create required arrays
```

```
op=[]; op=zeros(4, 1);
```

```
t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =  
1:length(t_01);
```

```
t_0S = []; func_0S = [];
```

```
% ep = 10^-6;
```

```
S=inline(['1 - ((x-' num2str(t1) ')/(' num2str(t2) '-' num2str(t1) '))']);  
Seval=feval(S,mval);  
Seval = double(Seval);  
t_0S = 0.000002:0.000002:Seval;  
func_0S = 1:length(t_0S);  
% figure integral value over [0,1]  
for k1 = 1:length(t_01)  
    func_01(k1) = exp(-t_01(k1)/c);  
end  
Q = trapz(t_01,func_01);  
Q = double(Q);  
K = 1/Q;
```

```

% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_0S)
    func_0S(k2) = exp(-t_0S(k2)/c);
end
geval = K*trapz(t_0S,func_0S);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['(1-(exp(-( 1 - ((x-' num2str(t1) '))/(' num2str(t2) '-
    ' num2str(t1) ')) )./' num2str(c) '))))/(1-(exp(-(1./' num2str(c) '))))')]);
xpec = t1 + quad(gS,t1,t2,ep);
mu0 = t1 + quad(S,t1,t2,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 9. Uniform distribution and PH distortion

function [op] = efunifph(a,mval,t1,t2,ep)
% efunifph(a,mval,t1,t2,ep)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the PH distortion
% as applied to the uniform distribution. The user enters values for a
% (the distortion parameter), mval (the undistorted median), t1 (the lower
% limit of the uniform distribution), t2 (the upper limit),
% and ep (the precision in the numerical integration, typically 10^-14).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

```

```

t_OS = []; func_OS = [];

% ep = 10^-6;

S=inline(['1 - ((x-' num2str(t1) ')/(' num2str(t2) '-' num2str(t1) '))']);
Seval=feval(S,mval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (t_01(k1))^(a-1);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = (t_OS(k2))^(a-1);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['( 1 - ((x-' num2str(t1) ')/(' num2str(t2) '-'
    ' num2str(t1) ')) ).^(' num2str(a) '.*1)']);
xpec = t1 + quad(gS,t1,t2,ep);
mu0 = t1 + quad(S,t1,t2,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 10. Weibull distribution and DP distortion

function [op] = efweibdp(b,mval,be1,t1,ep,N)

```



```

% efweibph(b,mval,be1,t1,ep,N)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the DP distortion
% as applied to the Weibull distribution. The user enters values for b
% (the distortion parameter), mval (the undistorted median), be1 (the beta
% parameter of the uniform distribution), t1 (the theta parameter),
% ep (the precision in the numerical integration, typically 10^-14), and
% N (the upper limit of the range of integration, typically 10^5 or more).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_OS = []; func_OS = [];

% ep = 10^-6;

S=inline(['exp((-1)*((x/' num2str(t1) ') .^' num2str(be1) '))']);
Seval=feval(S,mval);
Seval = double(Seval);
t_OS = 0.000002:0.000002:Seval;
func_OS = 1:length(t_OS);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (1-t_01(k1))^(b-1);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_OS)
    func_OS(k2) = (1-t_OS(k2))^(b-1);
end
geval = K*trapz(t_OS,func_OS);
geval = double(geval);
rat = geval/Seval;
% compute the expectation

```

```

gS = inline(['1-((1- exp((-1).*((x/' num2str(t1) ').^
' num2str(be1) ')) ))).^(' num2str(b) '.*1))']);
xpec = quad(gS,0,N,ep);
mu0 = quad(S,0,N,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
op(3) = Z;
op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 11. Weibull distribution and EX distortion

function [op] = efweibex(c,mval,be1,t1,ep,N)
% efweibph(c,mval,be1,t1,ep,N)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the EX distortion
% as applied to the Weibull distribution. The user enters values for c
% (the distortion parameter), mval (the undistorted median), be1 (the beta
% parameter of the uniform distribution), t1 (the theta parameter),
% ep (the precision in the numerical integration, typically 10^-14), and
% N (the upper limit of the range of integration, typically 10^5 or more).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t_01 = []; t_01 = 0.000002:0.000002:1.0; func_01 = []; func_01 =
1:length(t_01);

t_0S = []; func_0S = [];

% ep = 10^-6;

S=inline(['exp((-1).*((x/' num2str(t1) ').^' num2str(be1) '))']);
Seval=feval(S,mval);

```

```

        Seval = double(Seval);
        t_OS = 0.000002:0.000002:Seval;
        func_OS = 1:length(t_OS);
        % figure integral value over [0,1]
        for k1 = 1:length(t_O1)
            func_O1(k1) = exp(-t_O1(k1)/c);
        end
        Q = trapz(t_O1,func_O1);
        Q = double(Q);
        K = 1/Q;
        % figure integral value over [0,S(x)], where S(x)=Seval
        for k2 = 1:length(t_OS)
            func_OS(k2) = exp(-t_OS(k2)/c);
        end
        geval = K*trapz(t_OS,func_OS);
        geval = double(geval);
        rat = geval/Seval;
        % compute the expectation
        gS = inline(['(1-(exp(-( exp((-1).*(x/' num2str(t1) ')'.^
            ' num2str(be1) ')) )./' num2str(c) '))))/(1-(exp(-(1./' num2str(c) '))))')']);
        xpec = quad(gS,0,N,ep);
        mu0 = quad(S,0,N,ep);
        Z = xpec/mu0;
        op(1) = xpec;
        op(2) = rat;
        op(3) = Z;
        op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 12. Weibull distribution and PH distortion

function [op] = efweibph(a,mval,be1,t1,ep,N)
% efweibph(a,mval,be1,t1,ep,N)
% Author: Maj Edwin J. Offutt
% This m-file calculates efficiency and effectiveness of the PH distortion
% as applied to the Weibull distribution. The user enters values for a
% (the distortion parameter), mval (the undistorted median), be1 (the beta
% parameter of the uniform distribution), t1 (the theta parameter),
% ep (the precision in the numerical integration, typically 10^-14), and

```

% N (the upper limit of the range of integration, typically  $10^5$  or more).

format compact format long

% create required arrays

op=[]; op=zeros(4, 1);

t\_01 = []; t\_01 = 0.000002:0.000002:1.0; func\_01 = []; func\_01 =  
1:length(t\_01);

t\_0S = []; func\_0S = [];

% ep =  $10^{-6}$ ;

```
S=inline(['exp((-1)*((x/' num2str(t1) ') .^' num2str(be1) '))']);
Seval=feval(S,mval);
Seval = double(Seval);
t_0S = 0.000002:0.000002:Seval;
func_0S = 1:length(t_0S);
% figure integral value over [0,1]
for k1 = 1:length(t_01)
    func_01(k1) = (t_01(k1))^(a-1);
end
Q = trapz(t_01,func_01);
Q = double(Q);
K = 1/Q;
% figure integral value over [0,S(x)], where S(x)=Seval
for k2 = 1:length(t_0S)
    func_0S(k2) = (t_0S(k2))^(a-1);
end
geval = K*trapz(t_0S,func_0S);
geval = double(geval);
rat = geval/Seval;
% compute the expectation
gS = inline(['( exp((-1).*((x/' num2str(t1) ') .^'
    ' num2str(be1) ')) ) .^(' num2str(a) '.*1)']);
xpec = quad(gS,0,N,ep);
mu0 = quad(S,0,N,ep);
Z = xpec/mu0;
op(1) = xpec;
op(2) = rat;
```

```

        op(3) = Z;
        op(4) = Z/rat;

op

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The following four programs can be used to produce the distortion
% effects plots shown in Figures 4.6, 4.8, 4.10, and 4.12.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 1. Exponential distribution

function output = expo_plot_all4(a,b,c,lda,step,N)
% Author: Major Edwin J. Offutt
% User enters distortion parameter values of 0 < a <= 1, b >= 1,
% c >= 0, lambda (lda) > 0, step size for the numerical derivative (typically 0.05),
% and N (the upper limit of the range of integration, typically 10^5 or more).
% The m-file returns a plot of all four thesis distortions applied to the exponential
% distribution on a single set of axes, as well as the undistorted PDF.

    clc
    format compact
    format long

    % S=inline([num2str(lda) ' * exp((-1)*' num2str(lda) '*x)'])
    S=inline(['exp((-1)*' num2str(lda) '*x)']);

% ***** gamma beta
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
    and/or parameter "c" must be c >= 0') end
% a, b, c
X=[]; G=[]; for m = 0:step:N
    X = [X; m];
end K=1/quad(inline(['(x.^( ' num2str(a) '-1)).*((1-x).^( '
num2str(b) '-1)).*(exp(-x./' num2str(c) '))')],0,1); for n =
0:step:N
    z=feval(S,n);
    z=double(z);

```

```

        G=[G; K*quad(inline(['x.^(' num2str(a) '-1)').*((1-x).^(' num2str(b) '-1)).
            *(exp(-x./' num2str(c) '))']),0,z)];
    end X=double(X); G=double(G);
    % plot(X,Y);
    % hold on;

    Gpdf = [];

    for k = 2:length(G)
        Gpdf=[Gpdf; G(k-1)-G(k)];
    end

    Gpdf=Gpdf*20; plot(X(1:length(X)-1),Gpdf,'k'); hold on;

    % ***** proportional hazard
    % *****

    if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
        disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
            and/or parameter "c" must be c >= 0')
    end
    % a, b, c
    P=[];
    % for m = 0:step:N
    %     X = [X; m];
    % end
    K=1/quad(inline(['x.^(' num2str(a) '-1)']),0,1); for n = 0:step:N
        z=feval(S,n);
        z=double(z);
        P=[P; K*quad(inline(['x.^(' num2str(a) '-1)']),0,z)];
    end P=double(P);
    % plot(X,Y);
    % hold on;

    Ppdf = [];

    for k = 2:length(P)
        Ppdf=[Ppdf; P(k-1)-P(k)];
    end

    Ppdf=Ppdf*20; plot(X(1:length(X)-1),Ppdf, '--k');

```

```

% ***** dual power
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
D=[];
% for m = 0:step:N
%     X = [X; m];
% end
K=1/quad(inline(['(1-x).^(' num2str(b) '-1)']),0,1); for n =
0:step:N
    z=feval(S,n);
    z=double(z);
    D=[D; K*quad(inline(['(1-x).^(' num2str(b) '-1)']),0,z)];
end D=double(D);
% plot(X,Y);
% hold on;

Dpdf = [];

for k = 2:length(D)
    Dpdf=[Dpdf; D(k-1)-D(k)];
end

Dpdf=Dpdf*20; plot(X(1:length(X)-1),Dpdf, '-.k');

% ***** exponential
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
E=[];
% for m = 0:step:N
%     X = [X; m];
% end

```

```

K=1/quad(inline(['(exp(-x./' num2str(c) '))')'],0,1); for n =
0:step:N
    z=feval(S,n);
    z=double(z);
    E=[E; K*quad(inline(['(exp(-x./' num2str(c) '))')'],0,z)];
end E=double(E);
% plot(X,Y);
% hold on;

Epdf = [];

for k = 2:length(E)
    Epdf=[Epdf; E(k-1)-E(k)];
end

Epdf=Epdf*20; plot(X(1:length(X)-1),Epdf, 'k');

% ***** no distortion
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
    and/or parameter "c" must be c >= 0')
end
% a, b, c
M=[];
% for m = 0:step:N
%     X = [X; m];
% end
K=1/quad(inline(['(x.^0)']),0,1); for n = 0:step:N
    z=feval(S,n);
    z=double(z);
    M=[M; K*quad(inline(['(x.^0)']),0,z)];
end M=double(M);
% plot(X,Y);
% hold on;

Mpdf = [];

for k = 2:length(M)
    Mpdf=[Mpdf; M(k-1)-M(k)];
end

```



```

Mpdf=Mpdf*20; plot(X(1:length(X)-1),Mpdf, '--k*');

title('EXPONENTIAL DENSITY, solid=GB, ----=PH, -.-.=DP, ...=EX,
--*-=None')

% ***** create output to file
% *****

op = [X(1:length(X)-1) Gpdf Ppdf Dpdf Epdf Mpdf];

save MOut\expplot4.out op -ASCII

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 2. Triangular distribution

function output = tria_plot_all4(a,b,c,b1,c1,step,N)
% Author: Major Edwin J. Offutt
% User enters distortion parameter values of 0 < a <= 1, b >= 1,
% c >= 0, b1 (lower limit of the triangular distribution, c1 (mode
% for the triangular) step size for the numerical derivative (typically 0.05),
% and N (the upper limit of the range of integration, typically 10^5 or more).
% The m-file returns a plot of all four thesis distortions applied to the
% triangular distribution on a single set of axes, as well as the undistorted PDF.

    clc
    format compact
    format long

    a1=1;
    N=b1;

    S1=inline(['1-(((x-' num2str(a1) ')^2)/(( ' num2str(b1) '-' num2str(a1) '
    )*( ' num2str(c1) '-' num2str(a1) ')))']);
    S2=inline(['(( ' num2str(b1) '-x)^2)/(( ' num2str(b1) '-' num2str(a1) '
    )*( ' num2str(b1) '-' num2str(c1) ')))']);

% ***** gamma beta
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)

```

```

        disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
    end
    % a, b, c
    X=[]; G=[]; for m = 1:step:b1
        X = [X; m];
    end K=1/quadr(inline(['(x.^(' num2str(a) '-1)).*((1-x).^('
num2str(b) '-1)).*(exp(-x./' num2str(c) '))')'],0,1); for n =
1:step:c1
        z=feval(S1,n);
        z=double(z);
        G=[G; K*quadr(inline(['(x.^(' num2str(a) '-1)).*((1-x).^(' num2str(b) '-1))
        .*(exp(-x./' num2str(c) '))')'],0,z)];
    end for n = c1+step:step:b1
        z=feval(S2,n);
        z=double(z);
        G=[G; K*quadr(inline(['(x.^(' num2str(a) '-1)).*((1-x).^(' num2str(b) '-1))
        .*(exp(-x./' num2str(c) '))')'],0,z)];
    end X=double(X); G=double(G);
    % plot(X,Y);

    Gpdf = [];

    for k = 2:length(G)
        Gpdf=[Gpdf; G(k-1)-G(k)];
    end

    Gpdf=Gpdf*10; plot(X(1:length(X)-1),Gpdf, 'k'); hold on;

    % ***** proportional hazard
    % *****

    if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
        disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
    end
    % a, b, c
    P=[];
    % for m = 1:step:b1
    %     X = [X; m];
    % end
    K=1/quadr(inline(['(x.^(' num2str(a) '-1))')'],0,1); for n =
1:step:c1

```

```

        z=feval(S1,n);
        z=double(z);
        P=[P; K*quad(inline(['(x.^( ' num2str(a) '-1))']),0,z)];
    end for n = c1+step:step:b1
        z=feval(S2,n);
        z=double(z);
        P=[P; K*quad(inline(['(x.^( ' num2str(a) '-1))']),0,z)];
    end P=double(P);
    % plot(X,Y);

Ppdf = [];

for k = 2:length(P)
    Ppdf=[Ppdf; P(k-1)-P(k)];
end

Ppdf=Ppdf*10; plot(X(1:length(X)-1),Ppdf, '--k');

% ***** dual power
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
D=[];
% for m = 1:step:b1
%     X = [X; m];
% end
K=1/quad(inline(['((1-x).^( ' num2str(b) '-1))']),0,1); for n =
1:step:c1
    z=feval(S1,n);
    z=double(z);
    D=[D; K*quad(inline(['((1-x).^( ' num2str(b) '-1))']),0,z)];
end for n = c1+step:step:b1
    z=feval(S2,n);
    z=double(z);
    D=[D; K*quad(inline(['((1-x).^( ' num2str(b) '-1))']),0,z)];
end D=double(D);
% plot(X,Y);

Dpdf = [];

```

```

for k = 2:length(D)
    Dpdf=[Dpdf; D(k-1)-D(k)];
end

Dpdf=Dpdf*10; plot(X(1:length(X)-1),Dpdf, '-.k');

% ***** exponential
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
E=[];
% for m = 1:step:b1
%     X = [X; m];
% end
K=1/quad(inline(['(exp(-x./' num2str(c) '))'']),0,1); for n =
1:step:c1
    z=feval(S1,n);
    z=double(z);
    E=[E; K*quad(inline(['(exp(-x./' num2str(c) '))'']),0,z)];
end for n = c1+step:step:b1
    z=feval(S2,n);
    z=double(z);
    E=[E; K*quad(inline(['(exp(-x./' num2str(c) '))'']),0,z)];
end E=double(E);
% plot(X,Y);

Epdf = [];

for k = 2:length(E)
    Epdf=[Epdf; E(k-1)-E(k)];
end

Epdf=Epdf*10; plot(X(1:length(X)-1),Epdf, '-:k');

% ***** no distortion
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)

```

```

        disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
    end
    % a, b, c
    M=[];
    % for m = 1:step:b1
    %     X = [X; m];
    % end
    K=1/quadruple(integrate(@(x) x.^0),0,1); for n = 1:step:c1
        z=feval(S1,n);
        z=double(z);
        M=[M; K*quadruple(integrate(@(x) x.^0),0,z)];
    end for n = c1+step:step:b1
        z=feval(S2,n);
        z=double(z);
        M=[M; K*quadruple(integrate(@(x) x.^0),0,z)];
    end M=double(M);
    % plot(X,Y);

    Mpdf = [];

    for k = 2:length(M)
        Mpdf=[Mpdf; M(k-1)-M(k)];
    end

    Mpdf=Mpdf*10; plot(X(1:length(X)-1),Mpdf, '--k*');

    title('TRIANGULAR DENSITY, solid=GB, ----=PH, -.-.=DP, ....=EX,
    --*-=None')

    % ***** create output to file
    % *****

    op = [X(1:length(X)-1) Gpdf Ppdf Dpdf Epdf Mpdf];

    save Mout\triplot4.out op -ASCII

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % 3. Uniform distribution

    function output = unif_plot_all4(a,b,c,t2,step,N)

```

```

% Author: Major Edwin J. Offutt
% User enters distortion parameter values of  $0 < a \leq 1$ ,  $b \geq 1$ ,
%  $c \geq 0$ ,  $t_2$  (lower limit of the triangular distribution,
% step size for the numerical derivative (typically 0.05),
% and  $N$  (the upper limit of the range of integration, typically  $10^5$  or more).
% The m-file returns a plot of all four thesis distortions applied to the
% uniform distribution on a single set of axes, as well as the undistorted PDF.

clc
format compact
format long

% S=inline([num2str(lda) ' * exp((-1)*' num2str(lda) '*x)'])
S=inline(['1 - ((x - 1) / (' num2str(t2) '-1))'])];

% ***** gamma beta
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be  $0 \leq a \leq 1$ , parameter "b" must be  $b \geq 1$ ,
        and/or parameter "c" must be  $c \geq 0$ ')
end
% a, b, c
X=[]; G=[]; for m = 1:step:t2
    X = [X; m];
end K=1/quad(inline(['(x.^(' num2str(a) '-1)).*((1-x).^('
num2str(b) '-1)).*(exp(-x./' num2str(c) '))')'],0,1); for n =
1:step:N
    z=feval(S,n);
    z=double(z);
    G=[G; K*quad(inline(['(x.^(' num2str(a) '-1)).*((1-x).^(' num2str(b) '-1))
        .*(exp(-x./' num2str(c) '))')'],0,z)];
end X=double(X); G=double(G);
% plot(X,Y);

Gpdf = [];

for k = 2:length(G)
    Gpdf=[Gpdf; G(k-1)-G(k)];
end

Gpdf=Gpdf*10; plot(X(1:length(X)-1),Gpdf, 'k'); hold on;

```

```

% ***** proportional hazard
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
P=[];
% for m = 1:step:t2
%     X = [X; m];
% end
K=1/quad(inline(['(x.^(' num2str(a) '-1))']),0,1); for n =
1:step:N
    z=feval(S,n);
    z=double(z);
    P=[P; K*quad(inline(['(x.^(' num2str(a) '-1))']),0,z)];
end P=double(P);

Ppdf = [];

for k = 2:length(P)
    Ppdf=[Ppdf; P(k-1)-P(k)];
end

Ppdf=Ppdf*10; plot(X(1:length(X)-1),Ppdf, '--k');

% ***** dual power
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
D=[];
% for m = 1:step:t2
%     X = [X; m];
% end
K=1/quad(inline(['((1-x).^(' num2str(b) '-1))']),0,1); for n =
1:step:N
    z=feval(S,n);
    z=double(z);

```

```

        D=[D; K*quad(inline(['((1-x).^(' num2str(b) '-1))']),0,z)];
    end D=double(D);

    Dpdf = [];

    for k = 2:length(D)
        Dpdf=[Dpdf; D(k-1)-D(k)];
    end

    Dpdf=Dpdf*10; plot(X(1:length(X)-1),Dpdf, '-.k');

% ***** exponential
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >= 0')
end
% a, b, c
E=[];
% for m = 1:step:t2
%     X = [X; m];
% end
K=1/quad(inline(['(exp(-x./' num2str(c) '))']),0,1); for n =
1:step:N
    z=feval(S,n);
    z=double(z);
    E=[E; K*quad(inline(['(exp(-x./' num2str(c) '))']),0,z)];
end E=double(E);

    Epdf = [];

    for k = 2:length(E)
        Epdf=[Epdf; E(k-1)-E(k)];
    end

    Epdf=Epdf*10; plot(X(1:length(X)-1),Epdf, ':k');

% ***** no distortion
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,

```



```

        and/or parameter "c" must be c >= 0')
end
% a, b, c
M=[];
% for m = 1:step:t2
%     X = [X; m];
% end
K=1/quad(inline(['(x.^0)']),0,1); for n = 1:step:N
    z=feval(S,n);
    z=double(z);
    M=[M; K*quad(inline(['(x.^0)']),0,z)];
end M=double(M);

Mpdf = [];

for k = 2:length(M)
    Mpdf=[Mpdf; M(k-1)-M(k)];
end

Mpdf=Mpdf*10; plot(X(1:length(X)-1),Mpdf, '--k*');

title('UNIFORM DENSITY, solid=GB, ----=PH, -.-.=DP, ....=EX,
--*-=None')

% ***** create output to file
% *****

op = [X(1:length(X)-1) Gpdf Ppdf Dpdf Epdf Mpdf];

save Mlout\uniplot4.out op -ASCII

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% 4. Weibull distribution

function output = weib_plot_all4(a,b,c,t1,be1,step,N)
% Author: Major Edwin J. Offutt
% User enters distortion parameter values of 0 < a <= 1, b >= 1,
% c >= 0, theta (t1), beta (be1), step size for the numerical derivative (typically 0.05),
% and N (the upper limit of the range of integration, typically 10^5 or more).
% The m-file returns a plot of all four thesis distortions applied to the
% Weibull distribution on a single set of axes, as well as the undistorted PDF.

```

```

clc
format compact
format short g

xi0=0;
a1=t1;
c1=be1;
S=inline(['exp((-1)*((x-' num2str(xi0) ')/' num2str(a1) ')-' num2str(c1) '))']);

% ***** gamma beta
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >=0')
end
% a, b, c
X=[]; G=[]; for m = ceil(xi0):step:(ceil(xi0)+N)
    X=[X; m];
end K=1/quad(inline(['(x.^(' num2str(a) '-1)).*((1-x).^('
num2str(b) '-1)).*(exp(-x./' num2str(c) '))')'],eps,1); for n =
ceil(xi0):step:(ceil(xi0)+N)
    z=feval(S,n);
    if z < eps
        z = eps;
    end
    z=double(z);
    G=[G; K*quad(inline(['(x.^(' num2str(a) '-1)).*((1-x).^(' num2str(b) '-1))
        .*(exp(-x./' num2str(c) '))')'],eps,z)];
end X=double(X); G=double(G);
% plot(X,Y);
% hold on;

Gpdf = [];

for k = 2:length(G)
    Gpdf=[Gpdf; G(k-1)-G(k)];
end

Gpdf=Gpdf*10; plot(X(1:length(X)-1),Gpdf, 'k'); hold on;

```

```

% ***** proportional hazard
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >=0')
end
% a, b, c
P=[];
% for m = ceil(xi0):step:(ceil(xi0)+N)
%     X=[X; m];
% end
K=1/quad(inline(['(x.^(' num2str(a) '-1))']),eps,1); for n =
ceil(xi0):step:(ceil(xi0)+N)
    z=feval(S,n);
    if z < eps
        z = eps;
    end
    z=double(z);
    P=[P; K*quad(inline(['(x.^(' num2str(a) '-1))']),eps,z)];
end P=double(P);
% plot(X,Y);
% hold on;

Ppdf = [];

for k = 2:length(P)
    Ppdf=[Ppdf; P(k-1)-P(k)];
end

Ppdf=Ppdf*10; plot(X(1:length(X)-1),Ppdf, '--k');

% ***** dual power
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >=0')
end
% a, b, c
D=[];

```

```

% for m = ceil(xi0):step:(ceil(xi0)+N)
%     X=[X; m];
% end
K=1/quad(inline(['(1-x).^(' num2str(b) '-1)']),eps,1); for n =
ceil(xi0):step:(ceil(xi0)+N)
    z=feval(S,n);
    if z < eps
        z = eps;
    end
    z=double(z);
    D=[D; K*quad(inline(['(1-x).^(' num2str(b) '-1)']),eps,z)];
end D=double(D);
% plot(X,Y);
% hold on;

Dpdf = [];

for k = 2:length(D)
    Dpdf=[Dpdf; D(k-1)-D(k)];
end

Dpdf=Dpdf*10; plot(X(1:length(X)-1),Dpdf, '-.k');

% ***** exponential
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >=0')
end
% a, b, c
E=[];
% for m = ceil(xi0):step:(ceil(xi0)+N)
%     X=[X; m];
% end
K=1/quad(inline(['(exp(-x./' num2str(c) '))']),eps,1); for n =
ceil(xi0):step:(ceil(xi0)+N)
    z=feval(S,n);
    if z < eps
        z = eps;
    end
    z=double(z);
    E=[E; K*quad(inline(['(exp(-x./' num2str(c) '))']),eps,z)];

```

```

end E=double(E);
% plot(X,Y);
% hold on;

Epdf = [];

for k = 2:length(E)
    Epdf=[Epdf; E(k-1)-E(k)];
end

Epdf=Epdf*10; plot(X(1:length(X)-1),Epdf, 'k');

% ***** no distortion
% *****

if (a > 1) | (a <= 0) | (b < 1) | (c < 0)
    disp('ERROR: Parameter "a" must be 0 <= a <= 1, parameter "b" must be b >= 1,
        and/or parameter "c" must be c >=0')
end
% a, b, c
M=[];
% for m = ceil(xi0):step:(ceil(xi0)+N)
%     X=[X; m];
% end
K=1/quad(inline(['(x.^0)']),eps,1); for n =
ceil(xi0):step:(ceil(xi0)+N)
    z=feval(S,n);
    if z < eps
        z = eps;
    end
    z=double(z);
    M=[M; K*quad(inline(['(x.^0)']),eps,z)];
end M=double(M);
% plot(X,Y);
% hold on;

Mpdf = [];

for k = 2:length(M)
    Mpdf=[Mpdf; M(k-1)-M(k)];
end

Mpdf=Mpdf*10; plot(X(1:length(X)-1),Mpdf, '--k*');

```

```

title('WEIBULL DENSITY, solid=GB, ----=PH, -.-.=DP, ...=EX,
--*-=None')

% ***** create output to file
% *****

op = [X(1:length(X)-1) Gpdf Ppdf Dpdf Epdf Mpdf];

save Mlout\weiplot4.out op -ASCII

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

```

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